Introduction to jet and arc spaces

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Outline of the talk

1. Jets and arcs on algebraic varieties
2. Jets and arcs via higher derivations
3. The geometry of arc spaces
4. Differentials on the space of arcs

Jets and arcs on algebraic varieties

Geometric picture: convergent arcs on complex manifolds

Let X be a complex manifold with $\dim X = N$. A convergent arc is a holomorphic map $\alpha : U \to X$, where $U \subset \mathbb{C}$ is a small complex disk.

Choose local coordinates t for U and $x_1, \dots, x_N$ for X. Then α is given by

$$\alpha(t) = (x_1(t), \dots, x_N(t)) \in \mathbb{C}\{t\}^N.$$

For $n \geqslant 0$ the n-jet of α is obtained by truncation of $\alpha(t)$ at order n:

$$x_i(t) = \sum_{j=0}^{\infty} x_{i,j} t^j \longmapsto \mathrm{jet}_n(x_i(t)) = \sum_{j \leqslant n} x_{i,j} t^j.$$

Jets and arcs on algebraic varieties

Let K be a field of any characteristic. Let X be an algebraic variety over K and $n \in \mathbb{N}$.

Definition:

- An arc on X is a morphism $\alpha : \operatorname{Spec} K[[t]] \to X$.
- An n-jet on X is a morphism $\alpha_n : \operatorname{Spec} K[t]/(t^{n+1}) \to X$

Notation: $\operatorname{Spec} K[[t]] = \{0, \eta\}$ and $\operatorname{Spec} K[t]/(t^{n+1}) = \{0\}$, where 0 is the unique closed point and η the generic point of $K[[t]]$.

Truncation of arcs

Recall: the rings $K[t]/(t^{n+1})$ form an inverse system and $K[[t]] = \varprojlim_n K[t]/(t^{n+1})$.

For $n \in \mathbb{N}$ and $\alpha : \operatorname{Spec} K[[t]] \to X$ consider the n-th truncation of α

$$\operatorname{Spec} K[t]/(t^{n+1}) \longrightarrow \operatorname{Spec} K[[t]] \xrightarrow{\alpha} X ,$$

where the first map is induced by natural projection $K[[t]] \to K[t]/(t^{n+1})$.

Question: When does an n-jet arise as a truncation of an arc?

Lemma

Let X be a variety over K and $p \in X(K)$. Then X is smooth at p iff every jet α_n with $\alpha_n(0) = p$ can be lifted to an arc α on X.

Recall: X is formally smooth over K if for

$$\begin{array}{ccc} X & \longrightarrow & \operatorname{Spec} C/J \\ \downarrow & & \downarrow \\ \operatorname{Spec} K & \longrightarrow & \operatorname{Spec} C \end{array},$$

with $J \subset C$ nilpotent, there exists a diagonal arrow $X \to \operatorname{Spec} C$.

For the other direction reduce to the hypersurface case and use X smooth at p iff tangent cone equals tangent space at p.

Connection to the convergent case: Greenberg–Artin approximation

Theorem (Greenberg 1966, M. Artin 1969)

Let $f_1, \ldots, f_r \in \mathbb{C}[x_1, \ldots, x_N]$ and assume there exists solution $x_1(t), \ldots, x_N(t) \in \mathbb{C}[[t]]$ of

$$f_i(x_1(t), \ldots, x_N(t)) = 0, \; 1 \leqslant i \leqslant N. \qquad (\star)$$

Then, for every $c \in \mathbb{N}$, there exists a solution $x_1'(t), \ldots, x_N'(t) \in \mathbb{C}\{t\}$ of $(\star)$ such that

$$x_i'(t) - x_i(t) \in (t^c).$$

Remark: In fact, $x_i'(t)$ can be chosen to be algebraic power series.

Example: Arcs on affine space

Let $\mathbb{A}^N = \operatorname{Spec} K[x_1, \dots, x_N]$. An arc α on $\mathbb{A}^N$ corresponds to a map

$$\alpha^* : K[x_1, \dots, x_N] \to K[[t]]$$

Thus α is given by $x_i(t) := \alpha^*(x_i) \in K[[t]]$ for $1 \leqslant i \leqslant N$. Conversely, each choice of $x_i(t)$ gives map $K[x_1, \dots, x_N] \to K[[t]]$.

Write

$$x_i(t) = \sum_{j \in \mathbb{N}} x_{i,j} t^j.$$

$\rightsquigarrow$ Arcs on $\mathbb{A}^N$ are in bijection to points of

$$\mathbb{A}^\infty := \operatorname{Spec} K[x_{i,j} \mid 1 \leqslant i \leqslant N,\ j \in \mathbb{N}].$$

Example continued: Arcs on a hypersurface

Let $X \subset \mathbb{A}^N$ be given by $f \in K[x_1, \ldots, x_N]$. An arc on X is a solution $x(t) \in K[[t]]^N$ of

$$f(x(t)) = f(x_1(t), \ldots, x_N(t)) = 0. \qquad (\star)$$

Write $x_i(t) = \sum_{j=0}^{\infty} x_{i,j} t^j$ and expand in t:

$$f(x(t)) = \sum_{\ell \geqslant 0} F_n t^i, \; F_n \in K[x_{i,j} \mid 1 \leqslant i \leqslant N, \; 0 \leqslant j \leqslant n].$$

Thus $(\star)$ is equivalent to $F_n = 0$ for all $n \geqslant 0$ and arcs on X are in bijection with points on

$$X_\infty := \operatorname{Spec} K[x_{i,j}]/(F_n \mid n \in \mathbb{N}) \subset \mathbb{A}^\infty.$$

Example continued: Arc space of the cusp

Let $f = x^2 + y^3 \in K[x, y]$ and $\alpha^1(t) = \sum_{i \geqslant 0} x_i t^i$, $\alpha^2(t) = \sum_{i \geqslant 0} y_i t^i$.

Then $f(\alpha^1(t), \alpha^2(t)) = 0$ is equivalent to the system

$$
\begin{aligned}
F_0 &= x_0^2 + y_0^3 = 0 \\
F_1 &= 2x_0x_1 + 3y_0^2y_1 = 0 \\
F_2 &= x_1^2 + 2x_0x_2 + 3y_0y_1^2 + 3y_0^2y_2 = 0 \\
F_3 &= 2x_1x_2 + 2x_0x_3 + y_1^3 + 3y_0y_1y_2 + 3y_0^2y_3 = 0 \\
F_4 &= x_2^2 + 2x_1x_3 + 2x_0x_4 + 3y_1^2y_2 + 3y_0y_2^2 + 3y_0y_1y_3 + 3y_0^2y_4 = 0 \\
&\;\;\ldots
\end{aligned}
$$

The space of n-jets: formal definition

Definition: The n-th jet space X_n of X is the K-scheme representing the n-th jet functor

$$Y \mapsto \mathrm{Hom}_K(Y \times_K \operatorname{Spec} K[t]/(t^{n+1}), X).$$

In particular, for $Y = \operatorname{Spec} K$,

$$\mathrm{Hom}_K(\operatorname{Spec} K, X_n) \simeq \mathrm{Hom}_K(\operatorname{Spec} K[t]/(t^{n+1}), X).$$

For $m \geqslant n$ we have truncation maps $\pi_{m,n} : X_m \to X_n$ induced by $K[t]/(t^{m+1}) \to K[t]/(t^{n+1})$.

Clearly $X_0 = X$ and $\pi_n := \pi_{n,0} : X_n \to X$ is given by $\alpha_n \mapsto \alpha_n(0)$.

Example: 1-jets are tangent vectors

Assume that $X = \operatorname{Spec} R$. Then 1-jets α_1 correspond to maps $\alpha_1^* : R \to K[t]/(t^2)$. For $r \in R$ write

$$\alpha_1^*(r) = \varphi(r) + d(r)t.$$

Then $d \in \operatorname{Der}_K(R, K)$, where K is R-algebra via $\varphi : R \to K$. The derivation d corresponds to a tangent vector at the point $\alpha_1(0) = \ker \varphi$. Conversely, every such d gives a 1-jet on X.

$\rightsquigarrow X_1 = \operatorname{Spec}(\operatorname{Sym}_R \Omega_{R/K})$ is the total Zariski cotangent space.

The space of arcs: formal definition

The morphisms $\pi_{m,n} : X_m \to X_n$ form a projective system for $m > n$.

Definition: The arc space of X is $X_\infty := \varprojlim_n X_n$.

Remarks:

1. A priori not clear that X_∞ exists as a scheme (will see later).
2. We have

$$\mathrm{Hom}_K(\mathrm{Spec}\, K, X_\infty) \simeq \mathrm{Hom}_K(\mathrm{Spec}\, K[[t]], X).$$

3. If $\dim X > 0$, then X_∞ non-Noetherian of infinite Krull dimension.

Functorial properties of the arc space

By definition of X_n the arc space X_∞ represents the functor

$$Y \mapsto \mathrm{Hom}_K(Y \widehat{\times}_K \operatorname{Spec} K[[t]], X),$$

where $Y \widehat{\times}_K \operatorname{Spec} K[[t]]$ is the formal completion of $Y \times \mathbb{A}^1$ along $Y \times 0$.

If $X = \operatorname{Spec}(R)$, then by definition X_∞ represents

$$S \in \mathrm{Alg}_K \mapsto \mathrm{Hom}_K(\operatorname{Spec} S[[t]], X_\infty).$$

For non-affine X this still holds, but proof is hard (uses derived algebraic geometry, c.f. Bhatt).

Jets and arcs via higher derivations

Higher derivations of rings

Let R, S be K-algebras and $n \in \mathbb{N} \cup \{\infty\}$. A higher derivation $D : R \to S$ of order n is given by K-linear maps $D_i : R \to S$ for $i \leqslant n$ such that

1. $D_0 : R \to S$ is a K-algebra map.
2. The *higher Leibniz rules* hold; that is,

$$D_i(r_1 r_2) = \sum_{j+l=i} D_j(r_1) D_l(r_2).$$

Remark: For each higher derivation D its order 1-component D_1 is a usual derivation.

An example

Example: If $\mathrm{char}(K) = 0$ and $R = S = K[x]$, for $i \in \mathbb{N}$ set

$$D_i := \frac{1}{i!} \frac{d^i}{dx^i}.$$

Note: for $f(x) \in K[x]$ we have that $D_i(f(x))$ is the coefficient of t^i in the Taylor expansion of $f(x+t)$. Thus D is defined for $\mathrm{char}(K) = p > 0$. In fact, $D_p(x^p) = 1$, whereas $\frac{d}{dx} x^p = p x^{p-1} = 0$.

Fact: Every $D : R \to S$ of order $n \in \mathbb{N} \cup \{\infty\}$ corresponds to $\alpha_D : R \to S[t]/(t^n)$ resp. $R \to S[[t]]$ via

$$\alpha_D(r) = \sum_{i \leqslant n} D_i(r) t^i.$$

$\rightsquigarrow$ for $S = K$ the map α_D is just an n-jet resp. arc on $X = \mathrm{Spec}(R)$.

Definition: The n-th Hasse–Schmidt algebra $\mathrm{HS}_K^n(R)$ of R is defined as the quotient of

$$R[r^{(i)} \mid r \in R, 0 \leqslant i \leqslant n]$$

by the ideal generated by

$$\begin{aligned} r^{(i)} + s^{(i)} - (r+s)^{(i)}, \, r, s \in R, \\ c^{(i)}, \, c \in K, \\ (rs)^{(i)} - \sum_{j+l=i} r^{(j)} s^{(l)}, \, r, s \in R. \end{aligned}$$

Remark: we have an inclusion $R \to \mathrm{HS}_K^n(R)$ given by $r \mapsto r^{(0)}$.

Fact: For any K-algebra S

$$\mathrm{Hom}_K(\mathrm{HS}_K^n(R), S) \simeq \mathrm{Hom}_K(R, S[t]/(t^{n+1})).$$

for $n \in \mathbb{N}$, and

$$\mathrm{Hom}_K(\mathrm{HS}_K^\infty(R), S) \simeq \mathrm{Hom}_K(R, S[[t]]).$$

Thus, if $X = \mathrm{Spec}(R)$, then $X_n = \mathrm{Spec}(\mathrm{HS}_K^n(R))$ for $n \in \mathbb{N} \cup \{\infty\}$.

Definition: The universal higher derivation $\gamma : R \to \mathrm{HS}_K^n(R)[t]/(t^{n+1})$ is the map corresponding to $\mathrm{id}_{\mathrm{HS}_K^n(R)}$.

Fact: For $\varphi : R \to R'$ there exists maps $\mathrm{HS}_K^n(R) \to \mathrm{HS}_K^n(R')$ given by $r^{(i)} \mapsto (\varphi(r))^{(i)}$ for $n \in \mathbb{N} \cup \{\infty\}$.

The global case

Let X be a variety over K, not necessarily affine.

Fact (Vojta): For $n \in \mathbb{N} \cup \{\infty\}$ there exists a sheaf of $\mathcal{O}_X$-algebras $\mathrm{HS}^n_{X/K}$ such that for each affine open $U = \mathrm{Spec}(R)$ we have

$$\Gamma(U, \mathrm{HS}^n_{X/K}) \simeq \mathrm{HS}^n_K(R).$$

Then: $X_n = \underline{\mathrm{Spec}}_X(\mathrm{HS}^n_{X/K})$ is the relative Spec of $\mathrm{HS}^n_{X/K}$.

Remark: For $f : X \to Y$ we get morphisms $f_n : X_n \to Y_n$.

Warning: The universal arc $\gamma : X_\infty \widehat{\times}_K \mathrm{Spec}\, K[[t]] \to X$ is a morphism between proper formal schemes.

The geometry of arc spaces

A brief historical detour

The investigation of arc spaces in the context of algebraic geometry originated in the work of Nash in the 1960s, c.f. "'Arc structure of singularities".

The main idea is that the geometry of X_n and X_∞ is deeply related to singularities of X.

Independently, jet and arc spaces appeared implicitly in the works of Kolchin and Ribenboim on differential algebra.

Lemma

Let $f : X \to Y$ be étale. Then $X_n = X \times_Y Y_n$ for $n \in \mathbb{N}$.

Proof: Follows immediately from $f : X \to Y$ formally étale.

Lemma

If X is smooth and $\dim X = d$, then there exists covering by opens $U \subset X$ such that $\pi_n : X_n \to X$ restricts to $\pi_n^{-1}(U) = U \times \mathbb{A}^{dn} \to U$.

Proof: Since X smooth, there exists covering by opens $U \subset X$ and étale morphisms $U \to \mathbb{A}^d$. Then apply previous lemma and the fact that $(\mathbb{A}^d)_n = \mathbb{A}^d \times \mathbb{A}^{dn}$ for $n \in \mathbb{N}$.

Kolchin's Irreducibility theorem

Corollary

If X is smooth over K, then X_∞ irreducible.

Proof: Follows from X_n irreducible and $X_m \to X_n$ surjective.

Theorem (Kolchin)

Let X be a variety over a field K of characteristic 0. Then X_∞ is irreducible.

Theorem (Mustata)

Let X be a variety over $\mathbb{C}$. Then X_n is irreducible for all $n \geqslant 1$ iff X has at most rational singularities.

Lifting arcs along resolutions

Lemma

Let $f : Y \to X$ be a proper birational morphism such that f is an isomorphism over $Z \subset X$ closed. Then f_∞ gives rise to a bijection

$$Y_\infty \setminus (f^{-1}(Z))_\infty \to X_\infty \setminus Z_\infty.$$

Proof: For $\alpha \in X_\infty \setminus Z_\infty$ we have $\alpha(\eta) \in X \setminus Z \simeq Y \setminus f^{-1}(Z)$. Thus we get

$$\begin{array}{ccc} \operatorname{Spec} K((t)) & \xrightarrow{\alpha(\eta)} & Y \\ \downarrow & & \downarrow f \\ \operatorname{Spec} K[[t]] & \xrightarrow[\alpha]{} & X. \end{array}$$

Now apply valuative criterion of properness.

Sketch of proof of Kolchin's theorem

We sketch the proof of Kolchin's theorem by induction on $\dim X$.

Assume X is irreducible with $Z := \operatorname{Sing} X$ and let $f : Y \to X$ be a resolution of singularities. Sufficient to prove: $f_\infty : Y_\infty \to X_\infty$ is dominant. From before

$$Y_\infty \setminus (f^{-1}(Z))_\infty \simeq X_\infty \setminus Z_\infty.$$

Write $Z = \bigcup Z_i$ with Z_i irreducible; by induction $(Z_i)_\infty$ irreducible. By generic smoothness, there exists $U_i \subset Z_i$ dense open such that $f|_{f^{-1}(U_i)} : f^{-1}(U_i) \to U_i$ smooth. Then $(U_i)_\infty \subset f_\infty(Y_\infty)$ and thus $(Z_i)_\infty \subset \overline{f_\infty(Y_\infty)}$.

Counterexample in positive characteristics

Warning: Kolchin's theorem fails for $\mathrm{char}(K) > 0$.

Example: Consider $X = V(x^p - y^p z) \subset \mathbb{A}^3$ over K with $\mathrm{char}(K) = p$. A resolution of singularities is given by the normalization $f : Y := \mathbb{A}^2 \to X$, $(u, v) \mapsto (uv, v, u^p)$. Restricting f to $Z := \mathrm{Sing}(X) \simeq \mathbb{A}^1$ we get

$$f|_{f^{-1}(Z)} \colon \mathbb{A}^1 \to \mathbb{A}^1,\ u \mapsto u^p.$$

A generic arc on Z at 0 is of the form $\alpha(t) = (0, 0, z(t))$, with $\mathrm{ord}_t\, z(t) = 1$. Such an α cannot be lifted via f and it can be shown that $\alpha \notin \overline{f_\infty(Y_\infty) \setminus (f^{-1}(Z))_\infty}$.

Using similar arguments one can construct the Nash map. For surfaces X over $\mathbb{C}$:

$$\{\text{Irred. cpts. of } \pi_\infty^{-1}(\operatorname{Sing} X)\} \simeq \{\text{Exc. div. of } Y \to X\},$$

where $Y \to X$ is a minimal resolution. In his 1968 preprint Nash conjectured this is a bijection; it was fully proven only in 2012 by de Bobadilla and Pe Pereira.

This is only one example of the link between singularities of X and topological properties of X_∞.

Question: What about the singularities of X_∞ itself?

Differentials on the space of arcs

Smooth points in the arc space

Recall that X_∞ is infinite-dimensional if $\dim X > 0$, therefore the definitions of regular points does not make sense anymore.

Question: What are the smooth points in the arc space?

Candidate: $\alpha \in X_\infty$ with $\mathcal{O}_{X_\infty,\alpha}$ formally smooth over K.

Theorem (Bourqui, Sebag)

Let $\alpha \in X_\infty \setminus (\operatorname{Sing} X)_\infty$. Then $\mathcal{O}_{X_\infty,\alpha}$ formally smooth over K iff the unique formal branch containing $\alpha(\eta)$ is smooth.

Remark: Will see later that $\alpha \in (\operatorname{Sing} X)_\infty$ is not formally smooth.

Differentials on the space of arcs - Notation

Goal: Study the sheaf of differentials $\Omega_{X_\infty/K}$ of X_∞.

Notation: For a K-algebra R and $n \in \mathbb{N} \cup \{\infty\}$:

- $R[[t]]_n := R[t]/(t^{n+1})$ if $n \in \mathbb{N}$ and $R[[t]]_\infty := R[[t]]$.
- $R_n := \mathrm{HS}^n_K(R)$ the n-th Hasse–Schmidt algebra of R.
- $\gamma_n : R \to R_n[[t]]_n$ the map corresponding to the universal higher derivation $D := (D_i)_i$.
- For $r \in R$ set $r^{(i)} := D_i(r)$ and identify r with $r^{(0)}$. Then γ_n is given by
$$r \mapsto \sum_{i \geqslant 0} r^{(i)} t^i.$$

Differentials on the space of arcs - affine case

Theorem (de Fernex, Docampo)

Let $\Omega_{R/K}$ the module of Kähler differentials. For $n \in \mathbb{N} \cup \{\infty\}$

$$\Omega_{R_n/K} \simeq \Omega_{R/K} \otimes_R Q_n,$$

where

1. $Q_n := (R_n[[t]]_n)^{\vee} = \mathrm{Hom}_{R_n}(R_n[[t]]_n, R_n)$ *if* $n \in \mathbb{N}$, *and*
2. $Q_\infty := \varinjlim_n (R_\infty[[t]]_n)^{\vee}$.

Remark: Q_n is an R-module via $\gamma_n : R \to R_n[[t]]_n$.

Remark: Q_n is free of rank $(n+1)$ over R_n, whereas Q_∞ is free of infinite rank over R_∞. Note that $Q_\infty \not\simeq (R_\infty[[t]])^{\vee}$.

A formula for the Jacobian ideal of X_∞

Let $R = K[x_1, \ldots, x_N]/(f_1, \ldots, f_r)$. Then the formula for the Kähler differentials implies

$$\frac{\partial f_i^{(p)}}{\partial x_j^{(q)}} = D_{p-q}\left(\frac{\partial f_i}{\partial x_j}\right).$$

In particular: $\frac{\partial f_i^{(p)}}{\partial x_j^{(p)}} = \frac{\partial f_i}{\partial x_j}$, where we again identify x_i with $x_i^{(0)}$.

Example: $f = x^2 + y^3$ and $f^{(2)} = x_1^2 + 2x_0x_1 + 3y_0y_1^2 + 3y_0^2y_2$. Then

$$\frac{\partial f^{(2)}}{\partial y_1} = D_1(3y^2) = 6y_0y_1.$$

Differentials on the space of arcs - affine case pt. II

Theorem (de Fernex, Docampo)

Let $\Omega_{R/K}$ the module of Kähler differentials. For $n \in \mathbb{N} \cup \{\infty\}$

$$\Omega_{R_n/K} \simeq \Omega_{R/K} \otimes_R Q_n,$$

Remark: This formula implies in particular a weaker version of the Birational Transformation Rule of Denef and Loeser. More applications next time.

Observation: Left side parametrizes tangents on infinitesimal data of order n on R, while right side should be some "order n" operation on tangents on R.

Higher derivations of modules à la Ribenboim

Let R, S be two K-algebras and $D : R \to S$ a higher derivation of order $n \in \mathbb{N} \cup \{\infty\}$. Let $M \in \mathrm{Mod}_R$, $N \in \mathrm{Mod}_S$.

A higher derivation $\Delta : M \to N$ of order n is a collection of K-linear maps $\Delta_i : M \to N$ for $i \leqslant n$ satisfying

$$\Delta_i(r \cdot m) = \sum_{j+l=i} D_j(r)\Delta_l(m), \; r \in R, m \in M.$$

Fact (Ribenboim): There exists an R_n-module $\mathrm{HS}^n_{R/K}(M)$ parametrizing higher derivations, called the Hasse–Schmidt module.

Theorem 1 (de Fernex, Docampo; C., Narváez)

For $n \in \mathbb{N} \cup \{\infty\}$ we have $\mathrm{HS}^n_{R/K}(M) \simeq M \otimes_R Q_n$.

A comparison result

Theorem 2 (C., Narváez Macarro)

For a K-algebra R and an R-module M we have:

$$\operatorname{Sym}_{R_n} \operatorname{HS}^n_{R/K}(M) \simeq \operatorname{HS}^n_K(\operatorname{Sym}_R M).$$

In particular, for the "'mysterious" module Q_n we have

$$Q_n = \text{deg. 1 elements of } \operatorname{HS}^n_K(\operatorname{Sym}_R R)$$

Proof of $\Omega_{R_n/K} \simeq \Omega_{R/K} \otimes_R Q_n$ now follows from the easy fact that:

$$\operatorname{HS}^n_K(\operatorname{HS}^m_K(R)) \simeq \operatorname{HS}^m_K(\operatorname{HS}^n_K(R)),$$

for $m, n \in \mathbb{N} \cup \{\infty\}$.

Theorem 3 (de Fernex, Docampo; C., Narváez Macarro)

For $n \in \mathbb{N} \cup \{\infty\}$ the assignment $M \mapsto \mathrm{HS}^n_{R/K}(M)$ glues to give a functor

$$\mathrm{QCoh}(X) \to \mathrm{QCoh}(X_n).$$

In particular, for $n \in \mathbb{N}$ this restricts to give a functor

$$\mathrm{Vect}(X) \to \mathrm{Vect}(X_n).$$

Remark: Original construction by de Fernex and Docampo uses pullback to the universal arc, which for $n = \infty$ gives a sheaf over a formal scheme.

Question: Which bundles over X_n arise in the above way?

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