Worksheet 2. Equivalence relations.

1) Let $3\mathbb{Z} = \{3k \mid k \in \mathbb{Z}\}$. On $\mathbb{Z}$ we define the relation $m\mathcal{R}n \iff m - n \in 3\mathbb{Z}$. Show that this is an equivalence relation. Describe the equivalence classes and the quotient set.

2) Let $A = \{1, 2, 3, 5, 8, 13, 21, 34\}$ and consider the relation $a\,\mathcal{R}\,b \iff 3$ divides $b^2 - a^2$. Show that this is an equivalence relation and describe the equivalence classes.

3) For a fixed positive integer n, set $n\mathbb{Z} := \{nk \mid k \in \mathbb{Z}\}$. Define $x\mathcal{R}y \iff x - y \in n\mathbb{Z}$. Show that this is an equivalence relation. Describe the equivalence classes and the quotient set.

4) On $\mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$ consider the relation given by: $(m, n)\mathcal{R}(m', n') \iff m \cdot n' = m' \cdot n$. Show that this is an equivalence relation. Can you describe the equivalence classes and the quotient set?

5) On $\mathbb{Z} \times \mathbb{Z}$ consider the relation given by: $(m, n)\mathcal{R}(m', n') \iff m \cdot n' = m' \cdot n$. Is this an equivalence relation?

6) Consider the following relation on $\mathbb{R}$: $x\mathcal{R}y \iff x - y \in \mathbb{Z}$. Show that this is an equivalence relation and describe the quotient set.

7) Consider the following relation on $\mathbb{R}$: $x\mathcal{R}y \iff \lfloor x \rfloor = \lfloor y \rfloor$, where $\lfloor z \rfloor = max\{m \in \mathbb{Z} \colon m \leq z\}$ (is the integral part of z). Show that $\mathcal{R}$ is an equivalence relation and describe the quotient set.

8) Let M be the set of lines on the plane $\mathbb{R}^2$. Consider the relation on M given by:

$$r_1 \mathcal{R}\, r_2 \text{ if and only if } r_1 = r_2 \text{ ó } r_1 \text{ is parallel to } r_2.$$

Show that this is an equivalence relation. What is the equivalence class of the line $2x + 3y - 1 = 0$? Describe the quotient set by finding a set of numbers X and a bijection $g : \mathbb{R}^2/\mathcal{R} \to X$.

9) Let $f \colon \mathbb{R} \to \mathbb{R} : x \mapsto f(x) = x^2$. Consider the following relation on $\mathbb{R}$: $x\mathcal{R}y \iff f(x) = f(y)$. Show that $\mathcal{R}$ is an equivalence relation. Describe the quotient set.

10) Let X and Y be two sets and let $f \colon X \to Y$ be a function. Consider the following relation on X:

$$x\mathcal{R}y \iff f(x) = f(y).$$

Show that $\mathcal{R}$ defines an equivalence relation on X. Describe the quotient set. If f is bijective, what is this quotient set?

11) Let A be a set and let B be a non-empty subset of A. Consider the following relations on the set $\mathcal{P}(A)$:

(i) $X\mathcal{R}_1Y \iff X \cap B = Y \cap B$.

(ii) $X\mathcal{R}_2Y \iff X \cup B = Y \cup B$.

(iii) $X\mathcal{R}_3Y \iff X \setminus B = Y \setminus B$.

Decide whether these relations are equivalence relations or not; for those that are equivalence relations, describe the corresponding quotient sets.