Worksheet 1. Sets and functions.

1. Give an alternative description of the following sets:

   a) $\{x \in \mathbb{R} \mid x^2 - 5x + 6 = 0\}$ *b)* $\{x \in \mathbb{Z} \mid x^2 - 5x + 6 = 0\}$
   c) $\{x \in \mathbb{R} \mid x < 3\}$ *d)* $\{x \in \mathbb{N} \mid x < 3\}$
   e) $\{x \in \mathbb{N} \mid \exists y \in \mathbb{N}$ such that $y + 1 < x\}$ *f)* $\{x \in \mathbb{R} \mid x^2 + 2 = 0\}$
   g) $\{x \in \mathbb{R} \mid \exists y \in \mathbb{R}$ such that $x = y^2\}$ *h)* $\{x \in \mathbb{N} \mid \exists y \in \mathbb{N}$ such that $y < 5$ y $x = y^2\}$.

   Here $\mathbb{N} = \{0, 1, 2, 3, \dots\}$.

2. Let $S = \{a, b, c, d\}$, $T = \{1, 2, 3\}$ and $U = \{b, 2\}$. Decide which of the following expressions are correct and which ones are not.

(1)	$\{a\} \in S$	(2)	$a \in S$	(3)	$\{a, c\} \subseteq S$
(4)	$\varnothing \in S$	(5)	$\{a\} \subseteq \mathcal{P}(S)$	(6)	$\{\{a\}, \{a, b\}\} \in \mathcal{P}(S)$
(7)	$\{a, c, 2, 3\} \subseteq S \cup T$	(8)	$U \subseteq S \cup T$	(9)	$b \in S \cap U$
(10)	$\{b\} \subseteq S \cap U$	(11)	$\{1, 3\} \in T$	(12)	$\{1, 3\} \subseteq T$
(13)	$\{1, 3\} \in \mathcal{P}(T)$	(14)	$\{\varnothing\} \in \mathcal{P}(S)$	(15)	$\varnothing \in \mathcal{P}(S)$
(16)	$\varnothing \subseteq \mathcal{P}(S)$	(17)	$\{\varnothing\} \subseteq \mathcal{P}(S)$		

3. Let $S = \{1, 2, 3, 4, 5\}$, $T = \{3, 4, 5, 7, 8, 9\}$, $U = \{1, 2, 3, 4, 9\}$ and $V = \{2, 4, 6, 8\}$ be subsets of $\mathbb{N}$. Describe the following sets:

   (a) $S \cap U$ (b) $(S \cap T) \cup U$ (c) $(S \cup U) \cap V$ (d) $(S \cup V) \setminus U$ (e) $(U \cup V \cup T) \setminus S$
   (f) $(S \cup V) \setminus (T \cap U)$.

4. Show that the following equalities hold:

   (a) $(A \cup B)^c = A^c \cap B^c$ (b) $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$
   (c) $(A \cup B) \cap A = A$ (d) $(A \cap B) \cup A = A$

5. Compute the power set of the empty set; i.e., describe $\mathcal{P}(\varnothing)$.

6. Prove or disprove the following assertions: (1) $\mathcal{P}(A \cup B) = \mathcal{P}(A) \cup \mathcal{P}(B)$ (2) $\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B)$

7. Suppose that S are V are as in exercise 3. Describe the elements in $S \times V$. Observe that this is a subset of $\mathbb{N} \times \mathbb{N}$.

8. Let $S = \{a, b\}$, $T = \{a\}$, $V = \{1, 2\}$ and $U = \{1\}$. Compare the following sets:

   (a) $(S \times V) \setminus (T \times U)$ (b) $(S \setminus T) \times (V \setminus U)$.

9. Let A, B and C be three sets. Decide whether the following statements are true of false.

(i)	$A \setminus (B \cup C) = (A \setminus B) \cup (A \setminus C)$	(ii)	$card(A \cup B) = card(A \setminus B) + card(B \setminus A) + card(A \cap B)$
(iii)	$A \times (B \triangle C) = (A \times B) \triangle (A \times C)$	(iv)	$\mathcal{P}(A \setminus B) = \mathcal{P}(A) \setminus \mathcal{P}(B)$
(v)	$A \subseteq B \iff \mathcal{P}(A) \subseteq \mathcal{P}(B)$	(vi)	$A \setminus B = A \setminus C \Longrightarrow B = C$.

   For part (ii) assume that A and B are finite sets.

10. Prove that the following expressions define functions. Which of them are injective? Which ones are surjective? And bijective?

    (i) $f\colon \mathbb{N} \to \mathbb{N}$ $f(m) = m + 2$ (ii) $g\colon \mathbb{N} \to \mathbb{N}$ $g(n) = n(n+1)$
    (iii) $f\colon \mathbb{R} \to \mathbb{R}$ $f(x) = \sqrt{x^2+1}$ (iv) $f\colon \mathbb{Q} \to \mathbb{Q}$ $f(x) = x^2 + 4x$
    (v) $g\colon \mathbb{N} \to \mathbb{Q}$ $g(n) = n/(n+1)$ (vi) $g\colon \mathbb{Z} \to \mathbb{N}$ $g(n) = n^2$.

11. Consider the following functions:

$$\begin{aligned} i) &\quad f:\mathbb{R} \longrightarrow \mathbb{R}, \quad f(x) = x^3 + 1 \\ ii) &\quad f:\mathbb{Z} \longrightarrow \mathbb{Z}, \quad f(n) = 2n + 4 \\ iii) &\quad f:\mathbb{Q} \longrightarrow \mathbb{Q}, \quad f(x) = 2x + 4. \end{aligned}$$

    For each of them, describe: $Im(f)$ and $f^{-1}(0)$.

12. Let $a \in \mathbb{R}$ be non-zero. Prove that the function $f : \mathbb{R} \setminus \{a\} \longrightarrow \mathbb{R} \setminus \{a\}$, given by the expression $f(x) = \dfrac{ax}{x-a}$ is bijective and compute its inverse.

13. Decide whether the functions $f,\ g : \mathbb{Z} \longrightarrow \mathbb{Z}$ given below are injective, surjective or bijective:

$$f(n) = \begin{cases} n+1 & \text{if } n \text{ is even,} \\ 2n & \text{if } n \text{ is odd;} \end{cases} \qquad g(n) = \begin{cases} n/2 & \text{if } n \text{ is even,} \\ n+1 & \text{if } n \text{ is odd.} \end{cases}$$

14. Let $f : \mathbb{R} \to \mathbb{R}$ be the function:

$$f(x) = \begin{cases} x^3 & \text{si } x < 0, \\ x - 27 & \text{si } x \geq 0. \end{cases}$$

    Is f injective or surjective? Compute $f \circ f$.

15. For each part (a)-(d) find a function $f\colon \mathbb{N} \to \mathbb{N}$ which is:

    a) Injective but not surjective.

    b) Surjective but not injective.

    c) Bijective.

    d) Not injective and not surjective.

16. Let $f : \mathcal{U} \longrightarrow \mathcal{U}$ be a function and let $A, B \subseteq \mathcal{U}$. Decide whether the following statements are true or false:

    $i)$ $f(A) \cap f(B) = f(A \cap B)$. $ii)$ $f^{-1}(A) \cap f^{-1}(B) = f^{-1}(A \cap B)$.
    $iii)$ $f^{-1}(f(A)) = A$. $iv)$ $f^{-1}(A^c) = \left(f^{-1}(A)\right)^c$.

17. Let $f : \mathbb{R} \mapsto \mathbb{R}$ be the function $f(x) = x^3 - 3x$. Compute $f((0,2))$, $f([-1,3))$ and $f^{-1}((0,\infty))$.

18. Assume that $f : A \longrightarrow B$ and $g : B \longrightarrow C$ are bijective. Show that $g \circ f$ is also bijective and that

$$(g \circ f)^{-1} = f^{-1} \circ g^{-1}.$$

19. How many injective functions can be defined from the set $\{a, b, c\}$ into itself?

20. Assume A is a finite set with $card(A) = n$. How many injective functions can we define from A into A?