Worksheet 5. Matrices. Determinants. Systems of linear equations.

1) Compute A^2, A^3, A^4 for

$$a)\ A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \qquad b)\ A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Compute A^n for $n \in \mathbb{N}$. (*Hint*: try to guess a possible value for A^n and use induction to give a proof).

2) Let $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$. Compute the values $\alpha, \beta \in \mathbb{R}$ for which the equality $A^2 + \alpha A + \beta I_2 = 0$ holds, where I_2 and 0 are the identity matrix and the null matrix, respectively.

3) Show that the sum of two symmetric matrices is symmetric. Is the product of two symmetric matrices always a symmetric matrix?

4) We say that a matrix $A \in \mathcal{M}_n$ is *idempotent* if $A^2 = A$. Show that:

a) The matrix $\begin{pmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{pmatrix}$ is idempotent.

b) If A is idempotent, then $I_n - A$ is idempotent (where I_n is the identity matrix of order n).

c) If A is idempotent, then $(I_n - A)A = A(I_n - A) = 0$.

d) If A is idempotent and invertible then A is the identity matrix.

5) Compute the following determinants:

$$a) \begin{vmatrix} 4 & -2 & 5 & 1 \\ -4 & 1 & 0 & -1 \\ 2 & 1 & -1 & 1 \\ 1 & 0 & 1 & -2 \end{vmatrix} \quad b) \begin{vmatrix} -4 & 1 & 1 & 1 & 1 \\ 1 & -4 & 1 & 1 & 1 \\ 1 & 1 & -4 & 1 & 1 \\ 1 & 1 & 1 & -4 & 1 \\ 1 & 1 & 1 & 1 & -4 \end{vmatrix} \quad c) \begin{vmatrix} x & a & a & a \\ a & x & a & a \\ a & a & x & a \\ a & a & a & x \end{vmatrix} \quad d) \begin{vmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 2 & 3 & 4 & 5 \\ 3 & 3 & 3 & 4 & 5 \\ 4 & 4 & 4 & 4 & 5 \\ 5 & 5 & 5 & 5 & 5 \end{vmatrix}.$$

Solutions: a) 63; b) 0; c) $(x-a)^3(3a+x)$; d) 5.

6) Find the values of $\lambda \in \mathbb{R}$ that are solutions of the following equations:

$$a) \begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} = 0 \quad b) \begin{vmatrix} 3-\lambda & 5 \\ 0 & -\lambda \end{vmatrix} = 0 \quad c) \begin{vmatrix} 1-\lambda & 2 & 3 \\ 0 & 1-\lambda & 2 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0$$

$$d) \begin{vmatrix} 3-\lambda & 1 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 1 & 3-\lambda \end{vmatrix} = 0 \quad e) \begin{vmatrix} 1-\lambda & 0 & -1 \\ 1 & -1-\lambda & -1 \\ -2 & 3 & 2-\lambda \end{vmatrix} = 0.$$

7) Use Gauss Elimination to discuss and solve the following systems of linear equations:

$$a)\ \left.\begin{array}{r} x_2 - 3x_3 = -5 \\ 2x_1 + 3x_2 + 3x_3 = 7 \\ 4x_1 + 5x_2 - 2x_3 = 10 \end{array}\right\} \quad b)\ \left.\begin{array}{r} 3x_1 - 10x_2 - x_3 = -15 \\ 2x_1 + 2x_2 + 3x_3 = 6 \\ x_1 + 14x_2 + 7x_3 = -1 \end{array}\right\} \quad c)\ \left.\begin{array}{r} x_1 + x_2 + x_3 = 2 \\ -x_1 + x_2 + x_3 = 1 \\ -x_1 + 3x_2 + 3x_3 = 4 \end{array}\right\}$$

$$d)\ \left.\begin{array}{r} 2x_1 - 2x_2 + x_3 = 9 \\ 3x_1 - 5x_2 + 2x_3 = 4 \\ 3x_1 + 3x_2 - x_3 = 9 \end{array}\right\} \quad e)\ \left.\begin{array}{r} x_1 - 2x_2 + x_3 = 7 \\ 3x_1 + 2x_2 - x_3 = 1 \\ 2x_1 - 5x_2 + 2x_3 = 6 \end{array}\right\} \quad f)\ \left.\begin{array}{r} x_1 - 3x_2 + 2x_3 = 0 \\ -x_1 - 2x_2 + 2x_3 = 0 \\ 2x_1 - 2x_3 = 0 \end{array}\right\}$$

$$g)\ \left.\begin{array}{r} x + y + z + t = 0 \\ y - z = 5 \\ x + z + 2t = 1 \\ x + 2y = 0 \end{array}\right\} \quad h)\ \left.\begin{array}{r} x_1 + 2x_2 + 3x_3 = 2 \\ x_1 - x_2 + x_3 = 0 \\ x_1 + 3x_2 - x_3 = -2 \\ 3x_1 + 4x_2 + 3x_3 = 0 \end{array}\right\} \quad i)\ \left.\begin{array}{r} x_1 + 2x_2 + 3x_3 + 4x_4 = 0 \\ -x_1 + x_2 + 9x_4 = 0 \\ -x_1 - 3x_2 + 2x_3 + 3x_4 = 0 \\ -x_1 + 2x_2 - 5x_3 + 2x_4 = 0 \end{array}\right\}$$

Solution: a) $\left(\frac{67}{11}, -\frac{28}{11}, \frac{9}{11}\right)$; b) incompatible; c) $\left\{\left(\frac{1}{2}, \frac{3}{2} - \alpha, \alpha\right) | \alpha \in \mathbb{R}\right\}$; d) $(1, 13, 33)$; e) $(2, 8, 21)$; f) $(0, 0, 0)$; g) $(-8, 4, -1, 5)$; h) $(-1, 0, 1)$; i) $\{(59\alpha, -22\alpha, -17\alpha, 9\alpha) | \alpha \in \mathbb{R}\}$.

8) Each of the following systems of equations has two different independent terms. Solve them simultaneously using Gauss Elimination.

$$a)\ \left.\begin{array}{rrrrr|r} 2x_1 - 4x_2 & & & = & 10 & -8 \\ x_1 - 3x_2 & & + x_4 & = & -4 & -2 \\ x_1 & - x_3 & + 2x_4 & = & 4 & 9 \\ 3x_1 - 4x_2 & + 3x_3 & - x_4 & = & -11 & -15 \end{array}\right\} \quad b)\ \left.\begin{array}{rrrr|r} 2x_1 - 4x_2 & & = & 10 & -8 \\ x_1 - 3x_2 & & = & -4 & -2 \\ x_1 & - x_3 & = & 4 & 9 \\ 4x_1 - 7x_2 & - x_3 & = & 10 & -15 \end{array}\right\}$$

$$c)\ \left.\begin{array}{rrrrr|r} 2x_1 - 4x_2 & & & = & 10 & -8 \\ x_1 - 3x_2 & & + x_4 & = & -4 & -2 \\ x_1 & - x_3 & + 2x_4 & = & 4 & 9 \end{array}\right\}$$

Solution: a) $\left(\frac{97}{13}, \frac{16}{13}, -\frac{157}{13}, -\frac{101}{13}\right)$ y $(0, 2, -1, 4)$; b) $(23, 9, 19)$ and incompatible; c) $\{(23 + 2\alpha, 9 + \alpha, 19 + 4\alpha, \alpha) | \alpha \in \mathbb{R}\}$ y $\{(-8 + 2\alpha, -2 + \alpha, -17 + 4\alpha, \alpha) | \alpha \in \mathbb{R}\}$.

9) Use Gauss Elimination to compute, when possible, the inverses of the following matrices.

$$a)\ \begin{pmatrix} 1 & 3 & -2 \\ 2 & 5 & -3 \\ -3 & 2 & -4 \end{pmatrix}, \quad b)\ \begin{pmatrix} 1 & 3 & -2 \\ 2 & 5 & -3 \\ 3 & 8 & -5 \end{pmatrix}.$$

10) Discuss the following systems of linear equations in terms of the values of the parameter $a \in \mathbb{R}$:

$$a)\ \left.\begin{array}{rcrcrcr} ax & + & y & + & z & = & 1 \\ x & + & ay & + & z & = & a \\ x & + & y & + & az & = & a^2 \end{array}\right\} \quad b)\ \left.\begin{array}{rcrcrcr} 3x & - & y & & & = & ax \\ 5x & + & y & + & 2z & = & ay \\ & & 4y & + & 3z & = & az \end{array}\right\}$$

11) Discuss the following systems of linear equations in terms of the values of the parameters $a, b \in \mathbb{R}$:

$$\left.\begin{array}{rcrcrcr} 2x & - & ay & + & bz & = & 4 \\ x & & & + & z & = & 2 \\ x & + & y & + & z & = & 2 \end{array}\right\}$$