Worksheet 4. Congruences.

1) (i) Let p be a prime number. Show that if $p|ab$, then $p|a$ or $p|b$.

(ii) Let $p \in \mathbb{Z}$ with $p \geq 2$. Assume that p satisfies the following condition:

$$\text{If } p|ab \text{ with } a, b \in \mathbb{Z} \text{ then } p|a \text{ or } p|b.$$

Show that p is a prime number.

2) If $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_m^{\alpha_m}$ is the prime factorization of n, how many divisors does n have?

3) Let a, b, m be natural numbers with a and b coprime. Show that if $a|m$ and $b|m$ then $ab|m$. Find a counterexample to show that if a and b are not coprime then the previous statement does not hold in general.

4) Let $n \in \mathbb{N}$. Show that $\sqrt{n} \in \mathbb{Q} \Leftrightarrow \sqrt{n} \in \mathbb{N}$.

5) Consider m consecutive integers: n, $n+1$, $n+2, \dots, n+(m-1)$, with $m > 1$. Show that one, and only one of them, is divisible by m.

6) Find all the units in $\mathbb{Z}_7$ and find their multiplicative inverses.

7) Find all the units in $\mathbb{Z}_8$ and find their multiplicative inverses.

8) Find the inverses of $\overline{13}$ and $-\overline{15}$ in $\mathbb{Z}_{23}$ and in $\mathbb{Z}_{31}$.

9) Find all the solutions of the following equations; if there is no solution, say why.

a) $\overline{13}x = \overline{2}$ in $\mathbb{Z}_{23}$

b) $\overline{16}x = \overline{7}$ in $\mathbb{Z}_{100}$.

c) $\overline{6}x = -\overline{10}$ in $\mathbb{Z}_{26}$.

d) $\overline{15}x = \overline{10}$ in $\mathbb{Z}_{20}$.

10) How many units are there in $\mathbb{Z}_{9630}$? How many unites are there in $\mathbb{Z}_{101}$?

11) Compute the remainder after dividing 6^{234} by 13.

12) Compute the remainder after dividing 15^{2098} by 14.

13) Show that the integer $5^{31} - 5$ is a multiple of 7.

14) Compute the remainder after dividing $15002^{8003} + 11^8$ by 15.

15) Show that the integer $13^{232} - 15$ is a multiple of 11.

16) Show that 4 divides $9(3^{611} - 5^{25})$.