

Greedy algorithm for Kolmogorov diameter

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Outline of the talk

- 1 Motivation
- 2 Greedy selection
- 3 Direct estimates
- 4 Technical setup
- 5 Rates of decay
- 6 Banach space
- 7 Greedy selection is unrealistic

Reduced basis

Let $F \subset \mathbb{R}^d$ be a compact set e.g. $F = [0, 1]^d$ with $d = 50$ and let

$$D_\mu f = g \quad \mu \in E \quad (1)$$

be a family of uniformly elliptic PDE's.

For a given $\mu \in F$ solving (1) is time consuming. We want to prepare ourselves to do it fast – on line. Idea of reduced basis method: (end of XX-century)

We solve (1) for $\mu_1, \dots, \mu_n$ to get $f_{\mu_1}, \dots, f_{\mu_n}$ and for given μ we approximate the solution

$$f_\mu \sim \sum_{j=1}^n a_j f_{\mu_j}.$$

How to find basis elements?

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- 1 $f_1 = \operatorname{argmax}\{\|f\| : f \in \mathcal{K}\}$
- 2 Given $f_1, \dots, f_n$ we define $E_n = \operatorname{span}\{f_1, \dots, f_n\}$ and put $f_{n+1} = \operatorname{argmax}\{\operatorname{dist}(f, E_n) : f \in \mathcal{K}\}$

We want to estimate the performance of this procedure.

We define $\sigma_n(\mathcal{K}) = \sup_{f \in \mathcal{K}} \operatorname{dist}(f, E_n)$.

Recall the Kolmogorov width

$$d_n(\mathcal{K}) = \inf_F \sup_{f \in \mathcal{K}} \operatorname{dist}(f, F)$$

where F is a subspace of dimension $\leq n$.

Clearly always $d_n(\mathcal{K}) \leq \sigma_n(\mathcal{K})$.

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This talk reports

Results from two papers

- BCDDPW Peter Binev, Albert Cohen, Wolfgang Dahmen, Ronald DeVore, Guergana Petrova, P.W., *Convergence Rates for Greedy Algorithms in Reduced Basis Methods* SIAM J. Math. Anal. vol.43 N.3 pp. 1457-1472 (2011)
- DPW Ronald DeVore, Guergana Petrova, P.W., *Greedy Algorithms for Reduced Basis in Banach Spaces* submitted

Let us digress

Note that $f_n \in \mathcal{K}$. We define

$$\bar{d}_n(\mathcal{K}) = \inf_F \sup_{f \in \mathcal{K}} \text{dist}(f, F)$$

where F is a subspace of dimension $\leq n$ spanned by elements from $\mathcal{K}$.

Theorem (BCDDPW)

The following holds:

- (i) *For any compact set $\mathcal{F} \subset \mathcal{H}$ and any $n \geq 0$, we have $\bar{d}_n(\mathcal{F}) \leq (n+1)d_n(\mathcal{F})$.*
- (ii) *Given any $n > 0$ and $\epsilon > 0$, there is a set $\mathcal{F} \subset \mathcal{H}$ such that $\bar{d}_n(\mathcal{F}) \geq (n-1-\epsilon)d_n(\mathcal{F})$.*

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Direct comparison

A.Buffa, Y.Maday, A.T.Patera, C.Prud'homme, G.Turinici, 2012, have shown that for $\mathcal{K} \subset \mathcal{H}$

$$\sigma_n(\mathcal{K}) \leq Cn2^n d_n(\mathcal{K}).$$

We have

Theorem (BCDDPW)

Let $\mathcal{F}$ be an arbitrary compact set in a Hilbert space $\mathcal{H}$. For each $n = 1, 2, \dots$ we have

$$\sigma_n(\mathcal{K}) \leq \frac{2^{n+1}}{\sqrt{3}} d_n(\mathcal{K}). \quad (2)$$

For any $n > 0$, $\epsilon > 0$ there exists a set $\mathcal{K} = \mathcal{K}_{n,\epsilon}$ such that

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Matrix representation in $\mathcal{H}$

Algorithm produces $f_1, f_2, \dots$. Let $f_n^*, n = 1, 2, \dots$, be its Gram-Schmidt orthogonalization, so $P_n f = \sum_{i=1}^n \langle f, f_i^* \rangle f_i^*$, and in particular

$$f_i = P_i f_i = \sum_{j=1}^i a_{i,j} f_j^*, \quad a_{i,j} = \langle f_i, f_j^* \rangle, \quad j \leq i. \quad (3)$$

The lower triangular matrix $A := A(\mathcal{F}) := (a_{i,j})_{i,j=1}^\infty$, $a_{i,j} = 0, j > i$, contains all the information about the output of algorithm on $\mathcal{F}$. It satisfies **P1**: The diagonal elements of A satisfy $|a_{n,n}| = \sigma_n := \sigma_n(\mathcal{F})$.

P2: For every $m \geq n$ one has $\sum_{j=n}^m a_{m,j}^2 \leq \sigma_n^2$.

Each matrix with **P1** and **P2** is an output of a greedy algorithm on some $\mathcal{F}$ e.g. $\{(f_n)_n\}$.

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Main Estimate

Let $d = d_m(\mathcal{F})$ and let $N > m$. Let $A_N = [a_{ij}]_{i,j=1}^N$. We have $\det A_N = \prod_{j=1}^N \sigma_j$. Assume that there exists m dimensional space X such that each row of the matrix A_N is closer then $d = d_m$ to X in ℓ_2 norm.

$$\prod_{i=1}^N \sigma_i^2 \leq \left\{ \frac{N}{m} \right\}^m \left\{ \frac{N}{N-m} d^2 \right\}^{N-m}. \quad (4)$$

Proof of (4): Fix an orthonormal basis for $\mathbb{R}^N$ which first spans X . Let C denotes the matrix G written in this basis. We denote by $\mathbf{c}_j$, the j -th column of C . We have

$$\begin{aligned} \prod_{i=1}^N \sigma_i^2 &= (\det A_N)^2 = (\det C)^2 \leq \prod_{j=1}^m \|\mathbf{c}_j\|_{\ell_2}^2 \cdot \prod_{j=m+1}^N \|\mathbf{c}_j\|_{\ell_2}^2 \\ &\leq \left(\frac{1}{m} \sum_{j=1}^m \|\mathbf{c}_j\|_{\ell_2}^2 \right)^m \cdot \left(\frac{1}{N-m} \sum_{j=m+1}^N \|\mathbf{c}_j\|_{\ell_2}^2 \right)^{N-m} \end{aligned}$$

Theorem (DPW)

For the greedy algorithm in a Hilbert space $\mathcal{H}$ and for any compact set $\mathcal{F}$, we have the following inequalities between $\sigma_n := \sigma_n(\mathcal{F})_{\mathcal{H}}$ and $d_n := d_n(\mathcal{F})_{\mathcal{H}}$, for any $N \geq 0$, $K \geq 1$, and $1 \leq m < K$,

$$\prod_{i=1}^K \sigma_{N+i}^2 \leq \left\{ \frac{K}{m} \right\}^m \left\{ \frac{K}{K-m} \right\}^{K-m} \sigma_{N+1}^{2m} d_m^{2K-2m}. \quad (5)$$

(DPW)

For the greedy algorithm in a Hilbert space $\mathcal{H}$ and for any compact set $\mathcal{F}$ and $n \geq 1$, we have

$$\sigma_n(\mathcal{F}) \leq \sqrt{2} \min_{1 \leq m < n} d_m^{\frac{n-m}{n}}(\mathcal{F}). \quad (6)$$

In particular $\sigma_{2n}(\mathcal{F}) \leq \sqrt{2} \sqrt{d_n(\mathcal{F})}$, $n = 1, 2, \dots$

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Rates of decay II

(BCDDPW, DPW)

If $d_n(\mathcal{F}) \leq C_0 n^{-\alpha}$, $n = 1, 2, \dots$, then $\sigma_n(\mathcal{F}) \leq C_1 n^{-\alpha}$, $n = 1, 2, \dots$, with $C_1 := 2^{5\alpha+1} C_0$.

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If $d_n(\mathcal{F}) \leq C_0 e^{-c_0 n^\alpha}$, $n = 1, 2, \dots$, then $\sigma_n(\mathcal{F}) \leq \sqrt{2C_0} e^{-c_1 n^\alpha}$, $n = 1, 2, \dots$, where $c_1 = 2^{-1-2\alpha} c_0$,

Note that for $\alpha > 1$ the estimate (2) i.e. $\sigma_n(\mathcal{K}) \leq \frac{2^{n+1}}{\sqrt{3}} d_n(\mathcal{K})$ is better—asymptotically it gives $c_1 = c_0$. For $\alpha < 1$ this is better (it preserves α). For $\alpha = 1$ the choice depends on the constants.

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Banach spaces–Technical setup

In $\mathcal{X}$ we get vectors $f_j \in \mathcal{X}$. For each $j = 0, 1, \dots$, we let $\lambda_j \in \mathcal{X}^*$ be the linear functional of norm one that satisfies

$$\lambda_j(E_j) = 0, \quad \lambda_j(f_j) = \text{dist}(f_j, E_j)_X = \sigma_{j+1}.$$

We associate with the greedy procedure a lower triangular matrix $A = (a_{i,j})_{i,j=0}^\infty$ with

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So diagonal elements $a_{j,j} = \sigma_j = \text{dist}(f_j, E_j)_X$.

Each entry $a_{i,j}$ satisfies

$$|a_{i,j}| = |\lambda_j(f_i)| = |\lambda_j(f_i - g)| \leq \|\lambda_j\|_{X^*} \|f_i - g\| = \|f_i - g\|, \quad j < i,$$

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Estimates for Banach space I

Suppose that X is a Banach space. For greedy algorithm applied to a compact set $\mathcal{F}$ contained in the unit ball of $\mathcal{X}$, the following holds for $\sigma_n := \sigma_n(\mathcal{F})_X$ and $d_n := d_n(\mathcal{F})_X$, $n = 1, 2, \dots$,

(DPW)

For any such compact set $\mathcal{F}$ and $n \geq 1$, we have

$$\sigma_n \leq \sqrt{2} \min_{1 \leq m < n} n^{\frac{n-m}{2n}} \left\{ \sum_{i=1}^n \sigma_i^2 \right\}^{\frac{m}{2n}} d_m^{\frac{n-m}{n}}.$$

In particular $\sigma_{2\ell} \leq 2\sqrt{\ell d_\ell}$, $\ell = 1, 2, \dots$

Estimates for Banach space II

(DPW)

If for $\alpha > 0$, we have $d_n \leq C_0 n^{-\alpha}$, $n = 1, 2, \dots$, then for any $0 < \beta < \min\{\alpha, 1/2\}$, we have $\sigma_n \leq C_1 n^{-\alpha+1/2+\beta}$, $n = 1, 2, \dots$, with $C_1 = C_1(\alpha, \beta, C_0)$

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(iii) If for $\alpha > 0$, we have $d_n \leq C_0 e^{-c_0 n^\alpha}$, $n = 1, 2, \dots$, then $\sigma_n < \sqrt{2C_0} \sqrt{n} e^{-c_1 n^\alpha}$, $n = 1, 2, \dots$, where $c_1 = 2^{-1-2\alpha} c_0$. The factor $\sqrt{n}$ can be deleted by reducing the constant c_1 .

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Example

It is classical (**Kashin-Gluskin**) that for $\mathcal{F} = \{\pm e_j\}_{j=1}^n$ we have $d_s(\mathcal{F})_{\ell_n^\infty} \leq C \sqrt{\ln \frac{n}{s}} s^{-1/2}$ for $1 \leq s \leq n/2$. Obviously $\sigma_s(\mathcal{F}) = 1$ for $1 \leq s < n$. We use this to show

(DPW)

For any $\alpha > \frac{1}{2}$ there exists a compact set $\mathcal{F}_\alpha \subset c_0$ such that $\sigma_n(\mathcal{F}) \sim n^{-\alpha}$ and $d_n(\mathcal{F}_\alpha) \leq C n^{-\alpha-1/2}$.

Why greedy selection is unrealistic?

Let $\mathcal{K} =: \{f_\mu : \mu \in E\}$ comes from our PDE.

- We cannot compute $\max\{\|f\| : f \in \mathcal{K}\}$ because
 - ① It requires solving $D_\mu = f$ for all μ (or at least infinitely many).
 - ② The solution is a function in infinite dimensional space so we cannot compute its norm.
- Next steps even more problems.
- We cannot compute f_μ exactly.

Weak greedy algorithm

For fixed $\gamma \in (0, 1]$ we define **weak greedy algorithm** as

- 1 We choose f_1 so that $\|f_1\| \geq \gamma \max\{\|f\| : f \in \mathcal{F}\}$
- 2 Given $f_1, \dots, f_n$ we define $E_n = \text{span}\{f_1, \dots, f_n\}$ and put f_{n+1} so that $\|f_{n+1}\| \geq \gamma \max\{\text{dist}(f, E_n) : f \in \mathcal{F}\}$

In the context of (1) this can be achieved cheaply.

All above results hold for weak greedy algorithm but γ must be incorporated into the constant

Weak greedy algorithm

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Realistic scheme

We have one more problem: when we compute f_j we obtain some errors.

We have $\gamma \in (0, 1]$ and accuracy $\epsilon > 0$

- 1 At the first step we determine $f_1 \in \mathcal{F}$ such that $\|f_1\| \geq \gamma \sup_{f \in \mathcal{F}} \|f\|$. However, from computation we receive the noisy version $\hat{f}_1$ which is not even necessarily in $\mathcal{F}$. All we know about $\hat{f}_1$ is that

$$\|f_1 - \hat{f}_1\| \leq \epsilon.$$

- 2 Given $\hat{f}_1, \dots, \hat{f}_n$ we define $\hat{E}_n = \text{span}\{\hat{f}_1, \dots, \hat{f}_n\}$ and determine $f_{n+1} \in \mathcal{F}$ such that $\text{dist}(f_{n+1}, \hat{E}_n) \geq \gamma \sup_{f \in \mathcal{F}} \text{dist}(f, \hat{E}_n)$. Rather than f_{n+1} we receive the noisy version $\hat{f}_{n+1}$ for which we only know $\|f_{n+1} - \hat{f}_{n+1}\| \leq \epsilon$.

Let us define

$$\hat{\sigma}_n(\mathcal{F}) := \sup_{f \in \mathcal{F}} \text{dist}(f, \hat{E}_n),$$

which is the performance of this realistic scheme on $\mathcal{F}$.

Theorem (BCDDPW)

Suppose we are given $\gamma \in (0, 1]$ and $\epsilon > 0$. Suppose that

$$d_n(\mathcal{F}) \leq Mn^{-\alpha}, \quad n \geq 0, \quad (7)$$

for some M and $\alpha > 0$. Then,

$$\hat{\sigma}_n(\mathcal{F}) \leq C \max\{Mn^{-\alpha}, \epsilon\}, \quad n \geq 0, \quad (8)$$

with some $C = C(\alpha, \gamma, \epsilon)$.

Thank you for your attention

