Estimating the maximal singular integral by the singular integral

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El Escorial, June 2012

The problem

$$\int (T^* f)^2(x)\,dx \le c \int (Tf)^2(x)\,dx \quad ?$$

Context : smooth homogeneous convolution C-Z operators

$$Tf(x) = \text{p.v.} \int_{\mathbb{R}^n} f(x-y)\frac{\Omega(y)}{|y|^n}\,dy, \quad x \in \mathbb{R}^n.$$

Ω homogeneous of degree 0 and $\Omega \in C^\infty(S^{n-1})$

Cancellation:

$$\int_{|y|=1} \Omega(y)\,d\sigma(y) = 0$$

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The maximal singular integral:

$$T^* f(x) = \sup_{\varepsilon > 0} |T^\varepsilon f(x)|$$

$$T^\varepsilon f(x) = \int_{|y-x|>\varepsilon} f(x-y) \frac{\Omega(y)}{|y|^n} \, dy, \quad x \in \mathbb{R}^n.$$

Cotlar's inequality:

$$T^* f(x) \lesssim M(Tf)(x) + Mf(x), \quad x \in \mathbb{R}^n$$

$$\|T^* f\|_2 \lesssim \|Tf\|_2 + \|f\|_2 \lesssim \|f\|_2$$

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The Hilbert transform

$$Hf(x) = \text{p.v. } \frac{1}{\pi} \int_{-\infty}^{\infty} f(x-y) \frac{1}{y} \, dy, \quad x \in \mathbb{R}.$$

The Beurling transform

$$Bf(z) = \text{p.v. } -\frac{1}{\pi} \int_{\mathbb{C}} f(z-w) \frac{1}{w^2} \, dA(w), \quad z \in \mathbb{C}.$$

$$\frac{1}{z^2} = \frac{\overline{z}^2}{|z|^4} = \frac{\overline{z}^2}{|z|^2} \frac{1}{|z|^2}.$$

The Riesz transforms

$$R_j f(x) = \text{p.v.} \int f(x-y) \frac{y_j}{|y|^{n+1}} \, dy, \quad x \in \mathbb{R}^n.$$

Second order Riesz transforms

$$R_{j_k} f(x) = \text{p.v.} \int f(x-y) \frac{y_j y_k}{|y|^{n+2}} \, dy, \quad x \in \mathbb{R}^n, \quad j \neq k.$$

Higher order Riesz transforms

$$R_P f(x) = \text{p.v.} \int f(x-y) \frac{P(y)}{|y|^{n+d}} \, dy, \quad x \in \mathbb{R}^n,$$

P harmonic homogeneous polynomial of degree d.

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The problem for Hilbert and Beurling transforms

$$\int_{-\infty}^{\infty} (H^* f)^2(x)\, dx \leq c \int_{-\infty}^{\infty} f^2(x)\, dx = c \int_{-\infty}^{\infty} H f^2(x)\, dx.$$

$$B^{\varepsilon} f(z) = \frac{1}{|D(z,\varepsilon)|} \int_{D(z,\varepsilon)} Bf(w)\, dA(w)$$

$$B^* f(z) \leq M(Bf)(z), \quad z \in \mathbb{C}.$$

$$|\{z \in \mathbb{C} : B^* f(z) > t\}| \leq \frac{c}{t} \min\{\|f\|_1, \|Bf\|_1\}, \quad t > 0.$$

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First problematic case is R_j in $n \geq 2$.

Mateu-Verdera (2006): **It works**.

But the inequality

$$R_j^* f(x) \leq C\, M(R_j f)(x), \quad x \in \mathbb{R}^n$$

is false, even for $n = 1$.

Indeed, it is false that

$$|\{x \in \mathbb{R}^n : R_j^* f(x) > t\}| \leq \frac{c}{t} \|R_j f\|_1, \quad t > 0.$$

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An operator for which the L^2 control of T^* by T fails is

$$T = B + \lambda B^2 = B(I + \lambda B), \quad |\lambda| = 1,$$

Kernel of $B + B^2$ $$-\frac{1}{\pi}\frac{\overline{z}^2}{|z|^4} + \frac{2}{\pi}\frac{\overline{z}^4}{|z|^6}$$

Multiplier of $B + B^2$ $$\frac{\overline{\xi}}{\xi} + \frac{\overline{\xi^2}}{\xi^2} = \frac{\overline{\xi}\, 2\,\mathrm{Re}\,\xi}{\xi^2}$$

Question

How to describe those T for which T^* can be estimated by T in L^2 ?

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Question

How to describe those T for which T^* can be estimated by T in L^2 ?

A positive result: If T is an even higher Riesz transform then

$$T^* f(x) \leq C\, M(Tf)(x), \quad x \in \mathbb{R}^n.$$

If T is an odd higher Riesz transform then

$$T^* f(x) \leq C\, M^2(Tf)(x), \quad x \in \mathbb{R}^n,$$

where $M^2(Tf) = M(M(Tf))$.

The Calderón-Zygmund algebra A:

$$\lambda I + T, \quad \lambda \in \mathbb{R}, \quad T \text{ a smooth homogeneous C-Z operator.}$$

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Theorem (Mateu-Orobitg-V (2011))

TFAE *for* T *even*

(i) $T^* f(x) \leq C\, M(Tf)(x)$, $x \in \mathbb{R}^n$.

(ii) $\int |T^* f|^2 \leq C \int |Tf|^2$.

(iii) *If* $\Omega(x) = P_2(x) + P_4(x) + \cdots$, $|x| = 1$.

Then there exist an even harmonic homogeneous polynomial P*, such that* P/P_{2j} *for all* j *and* $T = R_P \circ U$*,* U *invertible in the* CZ *algebra* A.

Theorem (Mateu-Orobitg-Pérez-V (2010))

TFAE *for* T *odd*

(i) $T^* f(x) \le C\, M^2(Tf)(x)$, $x \in \mathbb{R}^n$

$$M^2 = M \circ M.$$

(ii) $\int |T^* f|^2 \le C \int |Tf|^2.$

(iii) *If* $\Omega(x) = P_1(x) + P_3(x) + \cdots$, $|x| = 1$.

Then there exist an odd harmonic homogeneous polynomial P, *such that* P/P_{2j+1} *and* $T = R_P \circ U$, U *invertible in the* CZ *algebra* A.

Pointwise estimate for even higher Riesz Transforms

T is an even higher Riesz transform and we want to prove

$$T^* f(x) \leq C\, M(Tf)(x), \quad x \in \mathbb{R}^n$$

which is equivalent to

$$\left| \int_{B^c} \frac{P(x)}{|x|^{n+d}} f(x)\, dx \right| = |T^1 f(0)| \leq C\, M(Tf)(0).$$

$$B = \{x \in \mathbb{R}^n : |x| \leq 1\}$$

Lemma

$$\frac{P(x)}{|x|^{n+d}}\chi_{B^c}(x) = Tb(x),$$

where b is a bounded function supported on B.

$$|T^1 f(0)| = \left|\int Tb(x) f(x)\,dx\right| = \left|\int b(x) Tf(x)\,dx\right|$$

$$\leq |B| \|b\|_\infty \frac{1}{|B|}\int_B |Tf(x)|\,dx.$$

Proof of the Lemma for $P(x) = x_1 x_2$.

$$\frac{x_1 x_2}{|x|^{n+2}} \chi_{B^c}(x) = Tb(x),$$

where b is a bounded function supported on B.

$$E(x) = c_n \frac{1}{|x|^{n-2}} \qquad \partial_1 \partial_2 E = c_n \; p.v. \frac{x_1 x_2}{|x|^{n+2}}$$

$$\varphi(x) = \begin{cases} E(x) & \text{on } B^c \\ A_0 + A_1 |x|^2 & \text{on } B \end{cases}$$

A_0 and A_1 chosen so that φ and $\nabla\varphi$ are continuous on ∂B.

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$$\partial_i \varphi = -c_n(n-2)\frac{x_i}{|x|^n} \quad \text{on } B^c, \qquad 2A_1 x_i \quad \text{on } B$$

$$\partial_1\partial_2 \varphi = c_1 \frac{x_1 x_2}{|x|^{n+2}} \quad \text{on } B^c, \qquad 0 \quad \text{on } B$$

$$\varphi = E * \Delta\varphi = E * c_2 \chi_B$$

$$c_1 \frac{x_1 x_2}{|x|^{n+2}} \chi_{B^c}(x) = \partial_1\partial_2\varphi = p.v. \frac{x_1 x_2}{|x|^{n+2}} * c_3 \chi_B = T(b).$$

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Remarks

For a second order Riesz transform S we have shown

$$S^{\epsilon}(f)(x) = \frac{1}{|B(x,\epsilon)|}\int_{B(x,\epsilon)} S(f)(y)\,dy$$

A weighted variant works for a general even higher Riesz transform.

$$\frac{1}{x}\chi_{(-1,1)^c}(x) = H(b).$$

Since $H(-H) = I$

$$b(x) = -H\left(\frac{1}{y}\chi_{(-1,1)^c}(y)\right)(x)$$

$$= \frac{1}{\pi}\int_{|y|>1}\frac{1}{y-x}\frac{1}{y}\,dy$$

$$= \frac{1}{\pi x}\log\frac{|1+x|}{|1-x|}.$$

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Difference between even and odd cases

Second order Riesz Transforms : differential operator $\triangle$

Even Riesz transform of order d: differential operator $(\triangle)^{d/2}$

Riesz Transform : pseudo differential operator $(-\triangle)^{1/2}$

Odd Riesz Transform of order d : pseudo differential operator $(-\triangle)^{d/2}$

Pseudo differential operators are non local : one looses the support and the boundedness of b. We have $b \in BMO(\mathbb{R}^n)$

Necessary conditions for $\|T^* f\|_2 \le C \, \|T f\|_2$

$$K(z) = -\frac{1}{\pi}\frac{xy}{|z|^4} + \frac{2}{\pi}\frac{P_4(z)}{|z|^6}$$

$$\widehat{p.v.\,K}(\xi) = \frac{uv}{|\xi|^2} + \frac{P_4(\xi)}{|\xi|^4}, \quad 0 \neq \xi = u + iv \in \mathbb{C}$$

$$E(z) = \frac{1}{8\pi}\,|z|^2 \,\log|z| \qquad \hat{E}(\xi) = |\xi|^{-4}$$

$$\left(\partial_1\partial_2\triangle + P_4(\partial_1,\partial_2)\right)(E) = p.v.K$$

$$\varphi(z) = \begin{cases} E(z) & \text{on } \mathbb{C} \setminus B \\ A_0 + A_1|z|^2 + A_2|z|^4 + A_3|z|^6 & \text{on } B \end{cases}$$

$$\varphi = E * \Delta^2 \varphi = E * b \qquad b = \Delta^2 \varphi = \chi_B(z)(\alpha + \beta|z|^2)$$

$$L = \partial_1 \partial_2 \triangle + P_4(\partial_1, \partial_2) \qquad L(\varphi) = L(E) * b = p.v.K * b = T(b)$$

$$L(\varphi) = \chi_{\mathbb{C}\setminus B}(z) K(z) + \chi_B(z)\, L(A_0 + A_1|z|^2 + A_2|z|^4 + A_3|z|^6).$$

$$L(A_0 + A_1|z|^2 + A_2|z|^4 + A_3|z|^6) = c\,xy.$$

$$T(b) = \chi_{\mathbb{C}\setminus B}(z)K(z) + cxy\chi_B(z)$$

$$cxy\chi_B(z) * f = T(f) * b - T^1(f),$$

$$\|cxy\chi_B(z) * f\|_2 \leq C\,\|T(f)\|_2, \quad f \in L^2(\mathbb{C}),$$

$$|\widehat{cxy\chi_B(z)}(\xi)| \leq C\, \frac{|uv|\xi|^2 + P_4(\xi)|}{|\xi|^4}, \quad \xi \neq 0.$$

$\widehat{\chi_B}(\xi) = J_1(\xi)/|\xi|$ $\qquad J_1$ the Bessel function

$$G_m(\xi) = J_m(\xi)/|\xi|^m \qquad \frac{1}{r}\frac{dG_m}{dr}(r) = -G_{m+1}(r)$$

$$\widehat{xy\chi_B(z)}(\xi) = -\partial_1\partial_2(G_1(|\xi|)) = -uv\, G_3(|\xi|)$$

$$|u\,v\,G_3(|\xi|)| \le C\,\frac{|uv|\xi|^2 + P_4(\xi)|}{|\xi|^4}$$

$$|u\,v\,G_3(1)| \le C\,|uv + P_4(\xi)| \qquad |\xi| = 1$$

$$P_4(\xi) = \mathsf{Re}(\lambda\xi^4) = \alpha(u^3v - v^3u) + \beta(u^4 + v^4 - 6u^2v^2)$$

$P_4(u,v) = \alpha(u^3v - v^3u)$ so that uv divides $P_4(\xi)$

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$$P_4(u,v) = \alpha(u^3v - v^3u) \qquad \text{so that} \quad uv \quad \text{divides} \quad P_4(\xi)$$

$$0 < |G_3(1)| \leq C\left(1 + \alpha(u^2 - v^2)\right) \qquad |\xi| = 1, \quad \text{so} \quad |\alpha| < 1$$

$$K(z) = \frac{xy}{|z|^4} + \frac{x^4 + y^4 - 6x^2y^2}{|z|^6}$$

$$K(z) = -\frac{1}{\pi}\frac{xy}{|z|^4} + \frac{2}{\pi}\frac{x^3y - x, y^3}{|z|^6}$$

Sufficient conditions for $\|T^* f\|_2 \leq C\, \|T f\|_2$

$$K(z) = -\frac{1}{\pi}\frac{xy}{|z|^4} + \frac{2}{\pi}\alpha\frac{x^3y - xy^3}{|z|^6} \qquad |\alpha| < 1$$

$$T(b)(z) = \chi_{\mathbb{C}\setminus B}(z)K(z) + cxy\chi_B(z) \qquad b = \chi_B(z)(\alpha + \beta|z|^2)$$

$$xy\chi_B(z) = T(\beta)(z), \qquad \beta \quad \text{bounded and} \quad |\beta(z)| \leq \frac{C}{|z|^3}$$

Step 1: $\quad xy\chi_B(x) = R(\beta_0)$

where $\quad R \quad$ is the Riesz transform with kernel $\quad -\dfrac{1}{\pi}\dfrac{xy}{|z|^4}$

β_0 is in $L^\infty(B)$ $\quad \beta_0$ satisfies a Lipschitz condition on B

and $\quad \displaystyle\int \beta_0(x)\,dx = 0$

Step 2 :

$$R(\beta_0) = RU(U^{-1}\beta_0) = T(U^{-1}\beta_0) = T(\beta)$$

$$\beta = U^{-1}\beta_0 = (\lambda I + V)\beta_0 = \lambda\beta_0 + V(\beta_0)$$

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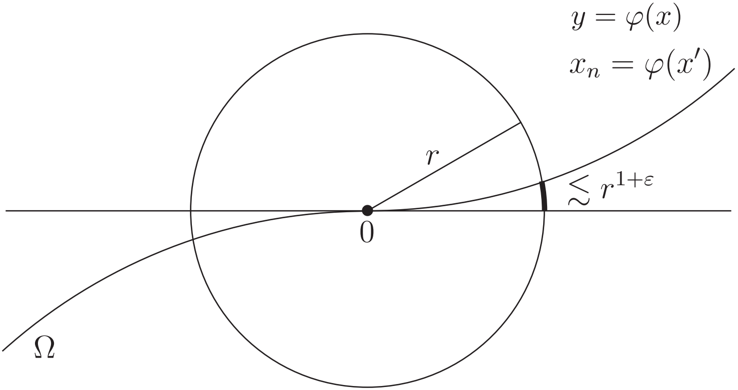
Lemma

If V is an even C-Z operator then $V(\chi_B) \in L^\infty(\mathbb{R}^n)$.



The Euler equation in the plane

$v(z,t)$ velocity field of an ideal incompressible fluid

$$(E) \quad \begin{cases} \partial_t v(z,t) + (v \cdot \nabla) v(z,t) = -\nabla p(z,t) \\ \nabla \cdot v = 0 \\ v(z,0) = v_0(z) \end{cases}$$

$$v \cdot \nabla = v_1 \partial_1 + v_2 \partial_2$$

Acceleration and force

Particle trajectory : $dz(t)/dt = v(z(t), t)$

$$\begin{aligned}\frac{d^2 z}{dt^2} &= \partial_t v(z(t), t) + \partial_1 \, v(z(t), t) \, v_1(z(t), t) + \dots \\ &= \partial_t v(z, t) + (v \cdot \nabla) v(z, t)\end{aligned}$$

Pressure and force

force on blob $V = \int_{\partial V} -p \, \vec{n} \, dS = \int_V -\nabla p \, dx$

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Pressure and force

$$\text{force on blob}\, V = \int_{\partial V} -p\, \vec{n}\, dS = \int_V -\nabla p\, dx$$

Incompressibility : the velocity field is divergence free

$$0 = \int_{\partial V} v \cdot \vec{n} \, dS = \int_V \operatorname{div}(v) \, dx$$

Vorticity

$$\omega = \operatorname{curl}(v) = \partial_1 v_2 - \partial_2 v_1$$

$$2\,\partial v = 2\,\frac{\partial}{\partial z} v = \operatorname{div} v + i \operatorname{curl} v = i\omega$$

$$\frac{1}{\pi \overline{z}} \quad \text{is the fundamental solution of} \quad \partial = \frac{\partial}{\partial z}$$

$$v(z,t) = \frac{i}{2}\frac{1}{\pi \overline{z}} * \omega = \frac{i}{2\pi} \int \frac{\omega(\zeta,t)}{\overline{z} - \overline{\zeta}}\, dA(\zeta)$$

How do you compute ∇v ?

$$\partial v = \frac{i}{2}\omega \qquad \text{and} \qquad \overline{\partial} v = -\frac{i}{2\pi}\, p.v.\frac{1}{\overline{z}^2} * \omega$$

If $\omega = \chi_D$ with D a bounded domain with smooth boundary,

then v is a Lipschitz field

The vorticity equation

$$\begin{cases} \partial_t \omega + (v \cdot \nabla)\omega = 0 \\ v = \dfrac{i}{2\pi} \dfrac{1}{\bar{z}} * \omega \\ \omega(z,0) = \omega_0(z) \end{cases}$$

Particle trajectory : $dz(t)/dt = v(z(t),t)$

$$\frac{d\omega(z(t),t)}{dt} = \partial_t \omega(z(t),t) + \partial_1 \omega(z(t),t) v_1(z(t),t) + \ldots$$

Equation of trajectories

$$\frac{dz(\alpha,t)}{dt} = v(z(\alpha,t),t) \qquad z(\alpha,0) = \alpha$$

$$\frac{dz(\alpha,t)}{dt} = \frac{i}{2\pi}\int \frac{\omega(\zeta,t)}{\bar{z}(\alpha,t)-\bar{\zeta}}\,dA(\zeta), \quad \zeta = z(\beta,t)$$

$$= \frac{i}{2\pi}\int \frac{\omega_0(\beta)}{\bar{z}(\alpha,t)-\bar{z}(\beta,t)}\,dA(\beta)$$

$$= f(z)(\alpha,t)$$

$$\frac{dz}{dt} = f(z)$$

$$\omega(z,t) = \omega(z(\alpha,t),t) = \omega_0(\alpha) \quad \text{if} \quad z = z(\alpha,t)$$

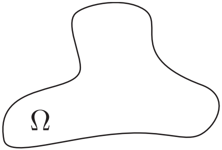
$$v(z,t) = \frac{i}{2\pi} \int \frac{\omega(\zeta,t)}{\bar{z} - \bar{\zeta}} \, dA(\zeta)$$

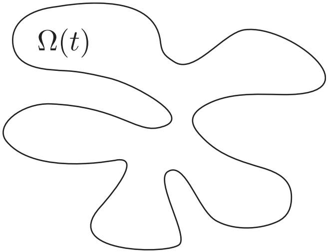
Yudovich's Theorem

If $\omega_0 \in L^\infty_C(\mathbb{C})$ there is a unique solution to the vorticity equation.

Vortex patches

$\omega_0 = \chi_\Omega$, Ω a domain $\qquad \omega(z,t) = \chi_{\Omega(t)}(z)$





Rotating vortex patches or V-states

Definition

A V-state is a vortex patch that rotates with constant angular velocity. If the initial domain D_0 has the origin as center of mass $D_t = e^{it\Omega} D_0$ for a certain angular velocity Ω

A disc rotates with any angular velocity

Kirchhoff : ellipses are V-states

$$\Omega = \frac{ab}{(a+b)^2}$$

Deem–Zabuski (1978) : numerical discovery of existence of V-states with m-fold symmetry

Burbea (1982) : analytical proof, by bifurcation

$$\psi(z,t) = \left(\frac{1}{2\pi}\log|z| * \chi_{D_t}\right)(z)$$

$$\psi(z,t) = \frac{\Omega}{2}|z|^2 + c, \quad z \in \partial D_t$$

$$\lambda|z|^2 + 2\,\mathrm{Re}\,\frac{1}{2\pi i}\int_{\partial D_t} \overline{\zeta}\log\left(1-\frac{z}{\zeta}\right)d\zeta = c, \quad z \in \partial D_t$$

Conformal mapping

$$\Phi(z) = z + a_o + \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots = z + f(z)$$

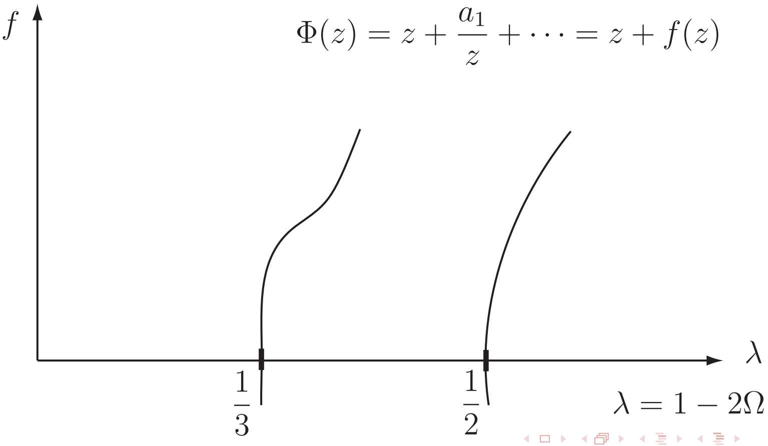
$$\lambda |\Phi(\omega)|^2 + S(\Phi)(\omega) = c, \quad |\omega| = 1$$

$$S(\Phi)(\omega) = \frac{1}{2\pi i} \int_{|\tau|=1} \overline{\Phi(\tau)} \log\left(1 - \frac{\Phi(\omega)}{\Phi(\tau)}\right) \Phi'(\tau)\, d\tau$$

Bifurcation

$$F(\lambda, f) = 0 \quad f \in C^{1+\alpha}(\mathbb{T})$$

$$\Phi(z) = z + \frac{a_1}{z} + \cdots = z + f(z)$$



Hmidi–Mateu–V : If the V–state is closed enough to the circle of bifurcation then the boundary is of class C^∞

$$\frac{\overline{\Phi'(\omega)}}{\Phi'(\omega)} = \omega^2 \, \frac{(1-\lambda)\overline{\Phi(\omega)} - I_1(\omega)}{(1-\lambda)\Phi(\omega) - \overline{I_1(\omega)}}, \qquad \lambda = \frac{1}{m}$$

$$I_1(\omega) = \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{\overline{\Phi(\tau) - \Phi(\omega)}}{\Phi(\tau) - \Phi(\omega)} \Phi'(\tau) d\tau$$

If $\partial\Omega_0$ is smooth, it is true that $\partial\Omega_t$ remains smooth for all $t > 0$?

Majda's Conjecture (1986)

There exists an initial smooth vortex patch which becomes of infinite length in finite time.

Chemin's Theorem (1993)

If $\partial\Omega_0 \in C^{1+\varepsilon}$ then $\partial\Omega_t \in C^{1+\varepsilon}$ for all $t > 0$.

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Thank you for your attention