Analysis on non-smooth domains

Tatiana Toro

University of Washington

9th International Conference in Harmonic Analysis & PDE

El Escorial, Madrid

Perturbation operators - Main tool

- Let L_1, L_0 be divergence form elliptic operators, the deviation function of L_1 from L_0 is

$$a(X) = \sup\{|A_1(Y) - A_0(Y)| : Y \in B(X, \delta(X)/2)\}$$

- Let Ω be a CAD, there is $\varepsilon_0 > 0$ so that if

$$\sup_{\Delta} \left(\frac{1}{\omega_0(\Delta)} \int_{T(\Delta)} a^2(X) \frac{G_0(X)}{\delta^2(X)} dX \right)^{1/2} < \varepsilon_0$$

then $\omega_1 \in B_2(\omega_0)$.

Why is this the main tool?

- Let Ω be a CAD and that assume $\omega_0 \in B_p(\sigma)$ for some $p > 1$. Given $\epsilon > 0$ there exists $\delta > 0$ such that if

$$\sup_{\Delta \subseteq \partial\Omega} \left\{ \frac{1}{\sigma(\Delta)} \int_{T(\Delta)} \frac{a^2(X)}{\delta(X)} dX \right\}^{1/2} \leq \delta,$$

then

$$\sup_{\Delta \subseteq \partial\Omega} \left\{ \frac{1}{\omega_0(\Delta)} \int_{T(\Delta)} a^2(X) \frac{G_0(X)}{\delta^2(X)} dX \right\}^{1/2} \leq \epsilon.$$

Thus $\omega_1 \in B_2(\omega_0)$ and $\omega_1 \in A_\infty(\sigma)$.

The main tool implies the perturbation result

- Assume $$\sup_{\Delta\subseteq\partial\Omega}\left\{\frac{1}{\sigma(\Delta)}\int_{T(\Delta)}\frac{a^2(X)}{\delta(X)}dX\right\}^{1/2}\leq C_0$$

- Consider $L_t=(1-t)A_0+tA_1$ for $0\leq t\leq 0$ and a partition of $[0,1]$ $\{t_i\}_{i=0}^m$ such that $0<t_{i+1}-t_i<\delta_0/C_0$. Let a_i be the deviation function of $L_{t_{i+1}}$ from L_{t_i}, $a_i(X)=(t_{i+1}-t_i)a(X)$.

- Then if δ_0 corresponds to ε_0 in the main tool we have

$$\sup_{\Delta\subseteq\partial\Omega}\left\{\frac{1}{\sigma(\Delta)}\int_{T(\Delta)}\frac{a_i^2(X)}{\delta(X)}dX\right\}^{1/2}\leq\delta_0\quad\&\quad\omega_{i+1}\in B_2(\omega_i)$$

- Iteration ensures that for $i\in\{0,\ldots,m\}$
  - $\omega_i\in A_\infty(\sigma)$ and $\omega_{i+1}\in B_2(\omega_i)$.

- Hence $\omega_1\in A_\infty(\sigma)$.

Sketch of the proof of the main tool

- Given $f \in C(\partial\Omega)$ consider for $i = 0, 1$

$$\begin{cases} L_i u_i = \operatorname{div}(A_i(X)\nabla u_i) &= 0 \text{ in } \Omega \\ u_i &= f \text{ on } \partial\Omega \end{cases}$$

- To show that $\omega_1 \in B_2(\omega_0)$ we need to show that

$$\|N(u_1)\|_{L^2(\omega_0)} \leq C\|f\|_{L^2(\omega_0)} \quad \text{where} \quad N(u_1) = \sup_{X \in \Gamma(Q)} |u_1(X)|$$

and

$$\Gamma(Q) = \{X \in \Omega : |X - Q| \leq 2\delta(X)\} \quad \text{and} \quad \delta(X) = \operatorname{dist}(\mathrm{X}, \partial\Omega)$$

- Strategy: Since L_1 is a perturbation of L_0 we view u_1 as a perturbation of u_0.

u_1 as a perturbation of u_0

Note that

$$u_1(X) = u_0(X) + \int_\Omega G_0(X,Y) L_0 u_1(Y) dY = u_0(X) + F(X),$$

where $G_0(X,\cdot)$ is the Green function of L_0 in Ω.

Integration by parts shows that

$$F(X) = \int_\Omega G_0(X,Y)(L_0 - L_1) u_1(Y) dY = \int_\Omega \nabla_Y G_0(X,Y) \varepsilon(Y) \nabla u_1(Y) dY$$

where $\varepsilon(Y) = A_1(Y) - A_0(Y)$.

Lemma 1. Let Ω be a CAD and assume

$$\sup_{\Delta}\left(\frac{1}{\omega_0(\Delta)}\int_{T(\Delta)} a^2(X)\frac{G_0(X)}{\delta^2(X)}dX\right)^{1/2} < \varepsilon_0$$

then

$$\|NF\|^2_{L^2(\omega_0)} + \|N(\delta|\nabla F|)\|^2_{L^2(\omega_0)} \lesssim \varepsilon_0^2\,\|S(u_1)\|^2_{L^2(\omega_0)}$$

where $S(u)$ denotes the square function of u given by

$$S^2(u)(Q) = \int_{\Gamma(Q)} |\nabla u(X)|^2\,\delta(X)^{2-n}dX.$$

Lemma 2. Let Ω be a CAD and assume

$$\sup_{\Delta} \left(\frac{1}{\omega_0(\Delta)} \int_{T(\Delta)} a^2(X) \frac{G_0(X)}{\delta^2(X)} dX \right)^{1/2} < \varepsilon_0$$

then

$$\|SF\|^2_{L^2(\omega_0)} \lesssim \left(\|N(\delta|\nabla F|)\|^2_{L^2(\omega_0)} + \|NF\|^2_{L^2(\omega_0)} + \|f\|^2_{L^2(\omega_0)} \right)$$

Proof of the theorem

Since $S(u_1) \le S(F) + S(u_0)$

$$\begin{aligned} \|NF\|^2_{L^2(\omega_0)} + \|N(\delta|\nabla F|)\|^2_{L^2(\omega_0)} &\lesssim \varepsilon_0^2 \, \|S(u_1)\|^2_{L^2(\omega_0)} \\ &\lesssim \varepsilon_0^2 \, \|SF\|^2_{L^2(\omega_0)} + \varepsilon_0^2 \, \|S(u_0)\|^2_{L^2(\omega_0)} \\ &\lesssim \varepsilon_0^2 \left[\|NF\|^2_{L^2(\omega_0)} + \|N(\delta|\nabla F|)\|^2_{L^2(\omega_0)} \right] \\ &\qquad + \varepsilon_0^2 \, \|f\|^2_{L^2(\omega_0)}, \end{aligned}$$

because

$$\|S(u_0)\|^2_{L^2(\omega_0)} \lesssim \|f\|^2_{L^2(\omega_0)}.$$

Thus for ε_0 small enough

$$\|NF\|^2_{L^2(\omega_0)} \lesssim \varepsilon_0^2 \, \|f\|^2_{L^2(\omega_0)}.$$

Recall our goal is to show

$$\|N(u_1)\|^2_{L^2(\omega_0)} \lesssim \|f\|^2_{L^2(\omega_0)}$$

Since $N(u_1) \leq N(F) + N(u_0)$,

$$\|N(u_0)\|^2_{L^2(\omega_0)} \lesssim \|f\|^2_{L^2(\omega_0)}$$

and

$$\|NF\|^2_{L^2(\omega_0)} \lesssim \varepsilon_0^2 \|f\|^2_{L^2(\omega_0)}$$

then we have

$$\|N(u_1)\|^2_{L^2(\omega_0)} \leq 2\left[\|NF\|^2_{L^2(\omega_0)} + \|N(u_0)\|^2_{L^2(\omega_0)}\right] \lesssim \|f\|^2_{L^2(\omega_0)}$$

which implies that $\omega_1 \in B_2(\omega_0)$.

Escauriaza's result

- Let Ω be a Lipschitz domain assume that

$$\lim_{r\to 0} \sup_{Q\in\partial\Omega} h(Q,r) = 0.$$

where

$$h(Q,r) = \left(\frac{1}{\sigma(\Delta(Q,r))} \int_{T(\Delta(Q,r))} \frac{a^2(X)}{\delta(X)} dX \right)^{1/2}.$$

If $\log k_0 \in VMO(\sigma)$ then $\log k_1 \in VMO(\sigma)$ where $k_j = \frac{d\omega_j}{d\sigma}$.

- Let Ω be a C^1 domain, $L_0 = \Delta$ and assume that

$$\lim_{r\to 0} \sup_{Q\in\partial\Omega} h(Q,r) = 0.$$

then $\log k_1 \in VMO(\sigma)$.

Motivating question

- Let Ω be a CAD with vanishing constant, $L_0 = \Delta$ and assume that

$$\lim_{r\to 0} \sup_{Q\in\partial\Omega} h(Q,r) = 0.$$

does $\log k_1 \in VMO(\sigma)$?

- How does this relate to the previous results?
  - $\log k \in VMO(\sigma)$ if and only if $\omega \in B_q(\sigma)$ for $q > 1$ and

$$\lim_{r\to 0} \sup_{Q\in\partial\Omega} \left(⨍_{B(Q,r)} k^q \, d\sigma \right)^{\frac{1}{q}} \left(⨍_{B(Q,r)} k \, d\sigma \right)^{-1} = 1$$

Lipschitz vs chord arc domains

- On a Lipschitz domain if $\log k_0 \in VMO(\sigma)$ and $\lim_{r\to 0}\sup_{Q\in\partial\Omega} h(Q,r) = 0$ then Dahlberg's result ensure that $\omega_1 \in B_2(\sigma)$. Escauriaza showed that and optimal B_2 inequality holds.
- What did we know?
- On a CAD if $\log k_0 \in VMO(\sigma)$ and $\lim_{r\to 0}\sup_{Q\in\partial\Omega} h(Q,r) = 0$ then $\omega_1 \in A_\infty(\sigma)$, i.e. $\exists q > 1$ such that $\omega_1 \in B_q(\sigma)$ (Milakis-Toro).
- Was this enough? NO

Lipschitz vs chord arc domains

- On a Lipschitz domain if $\log k_0 \in VMO(\sigma)$ and $\lim_{r\to 0}\sup_{Q\in\partial\Omega} h(Q,r) = 0$ then Dahlberg's result ensure that $\omega_1 \in B_2(\sigma)$. Escauriaza showed that and optimal B_2 inequality holds.
- On a CAD if $\log k_0 \in VMO(\sigma)$ and $\lim_{r\to 0}\sup_{Q\in\partial\Omega} h(Q,r) = 0$ then $\omega_1 \in A_\infty(\sigma)$, i.e. $\exists q > 1$ such that $\omega_1 \in B_q(\sigma)$ (Milakis-Toro).
- Results in [MPT] ensure that on a CAD if $\log k_0 \in VMO(\sigma)$ and $\lim_{r\to 0}\sup_{Q\in\partial\Omega} h(Q,r) = 0$ then $\omega_1 \in B_2(\sigma)$.

Regularity results for small perturbation operators - MPT

- Let Ω be a CAD if $\lim_{r\to 0} \sup_{Q\in\partial\Omega} h(Q,r) = 0$ and $\log k_0 \in VMO(\sigma)$ then

$$\left(\fint_{B(Q,r)} k_1^2 \, d\sigma \right)^{\frac{1}{2}} \left(\fint_{B(Q,r)} k_1 \, d\sigma \right)^{-1} \leq Cr^{\gamma} + Ch(Q,r)$$

$$+ \left(\fint_{B(Q,r)} k_0^2 \, d\sigma \right)^{\frac{1}{2}} \left(\fint_{B(Q,r)} k_0 \, d\sigma \right)^{-1}$$

In particular $\lim_{r\to 0} \sup_{Q\in\partial\Omega} h(Q,r) = 0$ and $\log k_0 \in VMO(\sigma)$ then $\log k_1 \in VMO(\sigma)$.

Sketch of the proof: Dahlberg's idea

For $t \in [0,1]$ consider the operators

$$\begin{aligned} L_t u &= \mathrm{div}(A_t \nabla u) \\ A_t(X) &= (1-t)A_0(X) + tA_1(X). \end{aligned}$$

Let ω_t be the elliptic measure of L_t and $k_t = \frac{d\omega_t}{d\sigma}$. For $Q \in \partial\Omega$ and $r > 0$ let $\Delta_r = B(Q,r) \cup \partial\Omega$. For $f \in L^2(\sigma)$ let

$$\Psi(t) = \frac{1}{\omega_t(\Delta_r)} \int_{\Delta_r} f\, k_t\, d\sigma.$$

Then $\Psi(t)$ is Lipschitz and

$$\dot{\Psi}(t) = \frac{1}{\omega_t(\Delta_r)} \int_{\Delta_r} \dot{k}_t \left(f - ⨍_{\Delta_r} f d\omega_t \right) d\sigma$$

where $\dot{k}_t$ is the weak L^2 limit of $(k_{t+h} - k_t)/h$ as h tends to zero.

Idea behind the proof

For $t \in [0,1]$ consider

$$\begin{aligned} L_t u_t &= \operatorname{div}(A_t \nabla u_t) \text{ in } \Omega \\ u_t &= f \text{ in } \partial\Omega \end{aligned}$$

For $t, s \in [0,1]$ and $\varepsilon(Y) = A_1(Y) - A_0(Y)$

$$u_s(X) - u_t(X) = (s-t) \int_\Omega \varepsilon(Y) \nabla G_t(X,Y) \nabla u_s(Y) dY.$$

$$\int_\Omega |\varepsilon(Y)| |\nabla G_t(X,Y)| |\nabla u_s(Y)| dY \lesssim ||f||_{L^2(\sigma)}$$

and

$$|u_s(X) - u_t(X)| \lesssim ||f||_{L^2(\sigma)} |s-t|.$$

Technical lemma

- There exist $\gamma, \beta \in (0,1)$ such that if $f \in L^2(\sigma)$, $f \geq 0$ and $\|f\|_{L^2(d\sigma/\sigma(\Delta_r))} \leq 1$ for $t \in [0,1]$

$$|\dot{\Psi}(t)| \leq C\left[r^\gamma + \sup_{s \leq r^\beta} \sup_{Q \in \partial\Omega} h(Q,s)\right]$$

- Integration guarantees that

$$\Psi(1) \leq \Psi(0) + C\left[r^\gamma + \sup_{s \leq r^\beta} \sup_{Q \in \partial\Omega} h(Q,s)\right]$$

- By duality

$$\frac{\sigma(\Delta_r)}{\omega_1(\Delta_r)}\left(\fint_{\Delta_r} k_1^2\, d\sigma\right)^{\frac{1}{2}} \leq \frac{\sigma(\Delta_r)}{\omega_0(\Delta_r)}\left(\fint_{\Delta_r} k_0^2\, d\sigma\right)^{\frac{1}{2}} + C\left[r^\gamma + \sup_{s \leq r^\beta} \sup_{Q \in \partial\Omega} h(Q,s)\right]$$

Open problems I

- Can a chord arc domain Ω be approximated by smooth interior chord arc domains $\Omega_m \subset \Omega$ in such a way that $\chi_{\Omega_m} \to \chi_\Omega$ in BV_{loc}?
  - A Lipschitz domain Ω can be approximated by smooth interior domains whose Lipschitz character is controlled by that Ω . Moreover the unit normal vector and the surface measure of the approximating domain converge to those of Ω.

Open problems II

- Hofmann-Mitrea-Taylor have studied the Neumann problem on chord arc domains with small and vanishing constant. Study the Neumann problem on general CADs.
  - [HMT] study the Neumann problem via layer potentials, using Semmes decomposition. This formulation of the Neumann problem is unavailable for general CADs.
- Study the regularity problem on CADs.