Analysis on non-smooth domains

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Outline

- Lecture I : The Dirichlet Problem
  - Divergence form elliptic operators (Laplacian)
  - Domains (NTA & CAD)
  - Boundary regularity for harmonic functions
- Lecture II: Harmonic Analysis on CAD
  - Perturbation operators
  - $(D)_p$ problem: a PDE question becomes a Harmonic Analysis problem
  - Tent spaces on chord arc domains

What do we know about general divergence form elliptic operators?

Recall we consider operators

$$Lu = \operatorname{div}(A(X)\nabla u)$$

where $A(X) = (a_{ij}(X))$ is symmetric measurable matrix and satisfies

$$\lambda|\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(X)\xi_i\xi_j \leq \Lambda|\xi|^2 \text{ for all } X \in \Omega \textit{ and } \xi \in \mathbb{R}^n.$$

- Caffarelli-Fabes-Kenig & Modica-Mortola constructed operators L on smooth domains for which ω_L and σ are mutually singular.

Question: *Characterize the operators L for which $\omega_L \in B_q(\sigma)$ for some $q \in (1, \infty)$.*

The behavior of A near $\partial\Omega$ determines the regularity of ω_L with respect to σ.

Remark: Let $L_0 = \mathrm{div}\,(A_0(X)\nabla)$ and $L_1 = \mathrm{div}\,(A_1(X)\nabla)$ be operators such that for $\ell = 0, 1$, $A_\ell(X) = (a^\ell_{ij}(X))$ is symmetric measurable matrix satisfying

$$\lambda|\xi|^2 \leq \sum_{i,j=1}^{n} a^\ell_{ij}(X)\xi_i\xi_j \leq \Lambda|\xi|^2 \text{ for all } X \in \Omega \textit{ and } \xi \in \mathbb{R}^n.$$

Assume A_0 and A_1 coincide in a neighborhood of $\partial\Omega$ then if $k_0 \in B_q(\sigma)$ then $k_1 \in B_q(\sigma)$.

- L_1 is a **perturbation** of L_0 if $L_1 = L_0$ on $\partial\Omega$ and $A_1(X)$ approaches $A_0(X)$ as X approaches the boundary in a *controlled* fashion.

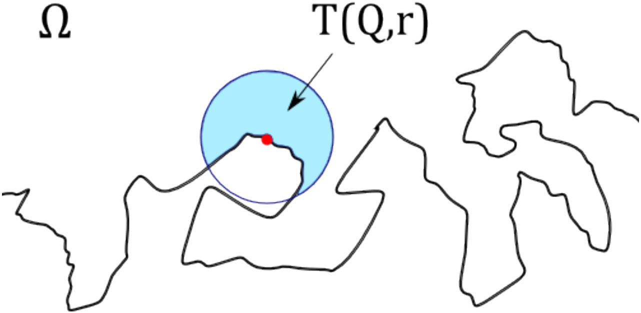
Perturbation operators

- L_1 is a **perturbation** of L_0 if the deviation function

$$a(X) = \sup\{|A_1(Y) - A_0(Y)| : Y \in B(X, \delta(X)/2)\}$$

satisfies $\frac{a^2(X)}{\delta(X)}\, dX$ is a Carleson measure.

Carleson regions



for all $Q \in \partial\Omega$ and $r \in (0, \operatorname{diam}\Omega)$,
$T(Q,r) = B(Q,r) \cap \Omega$ is the Carleson region associated to the surface ball $B(Q,r) \cap \partial\Omega$.

$\frac{a^2(X)}{\delta(X)}\,dX$ is a Carleson measure in Ω

if there exists $C > 0$ such that for all $Q \in \partial\Omega$ and $r > 0$

$$h(Q,r) = \left(\frac{1}{\sigma(B(Q,r))} \int_{T(Q,r)} \frac{a^2(X)}{\delta(X)}\,dX \right)^{1/2} < C$$

Note that in this case $A_1 = A_0$ on $\partial\Omega$.

Perturbation operators on Lipschitz domains

- Dahlberg: Assume

$$\lim_{r\to 0} \sup_{Q\in\partial\Omega} h(Q,r) = 0.$$

If $\omega_0 \in B_p(\sigma)$ for some $p > 1$ then $\omega_1 \in B_p(\sigma)$.

- R. Fefferman: Let

$$A(Q) = \left(\int_{\Gamma(Q)} \frac{a^2(X)}{\delta^n(X)} dX \right)^{1/2},$$

If $\omega_0 \in A_\infty(\sigma)$ and $\|A\|_{L^\infty(\sigma)} \le C$ then $\omega_1 \in A_\infty(\sigma)$.

General results for perturbation operators

- Fefferman-Kenig-Pipher: Let Ω be a Lipschitz domain. Assume

$$\sup_{Q\in\partial\Omega}\sup_{r>0} h(Q,r) < C$$

If $\omega_0 \in A_\infty(\sigma)$ then $\omega_1 \in A_\infty(\sigma)$.

- Milakis-Pipher-Toro: Let Ω be a chord arc domain. Assume

$$\sup_{Q\in\partial\Omega}\sup_{r>0} h(Q,r) < C$$

If $\omega_0 \in A_\infty(\sigma)$ then $\omega_1 \in A_\infty(\sigma)$.

Main Tool

- Fefferman-Kenig-Pipher. Let Ω be a Lipschitz domain, there is $\epsilon_0 > 0$ so that if

$$\sup_{\Delta} \left(\frac{1}{\omega_0(\Delta)} \int_{T(\Delta)} a^2(X) \frac{G_0(X)}{\delta^2(X)} dX \right)^{1/2} < \epsilon_0$$

then $\omega_1 \in B_2(\omega_0)$. Here $G_0(X)$ denotes the Green's function for L_0, $\Delta = B(Q,r) \cap \partial\Omega$ with $Q \in \partial\Omega$ and $r > 0$, and $T(\Delta) = B(Q,r) \cap \Omega$.

- Milakis-Pipher-Toro. Let Ω be a CAD, there is $\epsilon_0 > 0$ so that if

$$\sup_{\Delta} \left(\frac{1}{\omega_0(\Delta)} \int_{T(\Delta)} a^2(X) \frac{G_0(X)}{\delta^2(X)} dX \right)^{1/2} < \epsilon_0$$

then $\omega_1 \in B_2(\omega_0)$.

Why is this the main tool?

Theorem [MPT]: Let Ω be a CAD and let

$$A(a)(Q) = \left(\int_{\Gamma(Q)} \frac{a^2(X)}{\delta(X)^n} dX \right)^{\frac{1}{2}}$$

If $\|A(a)\|_{L^\infty(\sigma)} \leq C_0 < \infty$ and $\omega_0 \in A_\infty(\sigma)$ then $\omega_1 \in A_\infty(\sigma)$.

Proof: Fubini & properties of G_0 and ω_0 on NTA domains yield

$$\frac{1}{\omega_0(\Delta)} \int_{T(\Delta)} \frac{a^2(X) G_0(X)}{\delta(X)^2} dX \leq \frac{1}{\omega_0(\Delta)} \int_{3\Delta} A^2(a)(Q) d\omega_0(Q)$$

- $\exists \delta_0 > 0$ such that if $\|A(a)\|_{L^\infty(\sigma)} \leq \delta_0$ and $\omega_0 \in A_\infty(\sigma)$ then $\|A(a)\|_{L^\infty(\omega_0)} \leq \delta_0$, $\omega_1 \in B_2(\omega_0)$ and $\omega_1 \in A_\infty(\sigma)$.

- Consider $L_t = (1-t)A_0 + tA_1$ for $0 \leq t \leq 0$ and a partition of $[0,1]$ $\{t_i\}_{i=0}^m$ such that $0 < t_{i+1} - t_i < \delta_0 / C_0$. Let a_i be the deviation function of $L_{t_{i+1}}$ from L_{t_i}, $a_i(X) = (t_{i+1} - t_i)a(X)$.

- $\|A(a_i)\|_{L^\infty(\sigma)} = (t_{i+1} - t_i)\|A(a)\|_{L^\infty(\sigma)} < \delta_0$.

  Iteration ensures that for $i \in \{0, \ldots, m\}$
    - $\omega_i \in A_\infty(\sigma)$
    - $\omega_{i+1} \in B_2(\omega_i)$.

  Hence $\omega_1 \in A_\infty(\sigma)$.

Lipschitz domains vs chord arc domains

Differences & difficulties:

- Lack of a local representation for the boundary of CAD as a graph.
- Lack of results concerning approximation of CAD by interior smooth CAD.

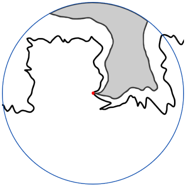
Tools:

- Estimate for the Green function on NTA domains.
- The geometry of the NTA domain and the Ahlfors regularity of the surface measure to the boundary ensure that integration over Carleson regions can be handled by Fubini as it was in half space (GMT).
- Harmonic analysis in CAD: We developed the theory of tent spaces for functions defined on CAD. Tent spaces for functions defined on half space were initially studied by Coifman-Meyer-Stein.

Non-tangential maximal functions on CAD

Recall

$$\Gamma(Q) = \{X \in \Omega : |X - Q| \leq 2\delta(X)\} \ \text{ with } \ \delta(X) = \mathrm{dist}\,(\mathrm{X}, \partial\Omega).$$



Define

$$N(f)(Q) = \sup\{|f(X)| : X \in \Gamma(Q)\}$$

$$\mathcal{N} = \{f : \Omega \to \mathbb{R} \ \text{ Borel } : N(f) \in L^1(\sigma)\},$$

$$\|f\|_{\mathcal{N}} = \|N(f)\|_{L^1(\sigma)}.$$

Carleson measures on CAD

- For a Borel measure μ on Ω and $Q \in \partial\Omega$ define

$$C(\mu)(Q) = \sup_{Q \in \Delta} \frac{1}{\sigma(\Delta)} \int_{T(\Delta)} d\mu,$$

where $\Delta = B(P, r) \cap \partial\Omega$, $P \in \partial\Omega$ and $r > 0$.

The set of Carleson measures in Ω is defined by:

$$\mathcal{C} = \{\mu \text{ Borel } : \|\mu\|_{\mathcal{C}} = \|C(\mu)\|_{L^\infty(\sigma)} < \infty\}$$

Duality between $\mathcal{N}$ and $\mathcal{C}$ on CAD

There exists a constant $C > 1$ such that if $f \in \mathcal{N}$ and $\mu \in \mathcal{C}$ then

$$\left| \int_{\Omega} f(X)\, d\mu(X) \right| \leq C \int_{\partial\Omega} N(f)(Q) C(\mu)(Q)\, d\sigma(Q) \leq \|f\|_{\mathcal{N}} \|\mu\|_{\mathcal{C}}$$

Tools:

- For $\mathcal{F} \subset \partial\Omega$ closed and $\alpha > 0$

$$\mathcal{R}_\alpha(\mathcal{F}) = \bigcup_{Q \in \mathcal{F}} \{X \in \Omega : |X - Q| \leq (1+\alpha)\delta(X)\} = \bigcup_{Q \in \mathcal{F}} \Gamma_\alpha(Q)$$

- For $\mathcal{O} \subset \partial\Omega$ open with $\mathcal{O} \cup \mathcal{F} = \partial\Omega$, the tent over $\mathcal{O}$ is $T(\mathcal{O}) = \Omega \backslash \mathcal{R}_\alpha(\mathcal{F})$.
- A Whitney type decomposition of $\mathcal{O}$ and Ahlfors regularity of σ guarantee that if μ is a Carleson measure on Ω then

$$\mu(T(\mathcal{O})) \lesssim \sigma(\mathcal{O})$$

Important inequalities

For $\mathcal{F} \subset \partial\Omega$ closed, $\alpha > 0$ and $\gamma \in (0,1)$,

$$\mathcal{R}_\alpha(\mathcal{F}) = \bigcup_{Q \in \mathcal{F}} \Gamma_\alpha(Q)$$

$$\mathcal{F}_\gamma^* = \{Q \in \partial\Omega : \ \sigma(\Delta \cap \mathcal{F}) \geq \gamma\sigma(\Delta) : \ \text{for all } \Delta = \Delta(Q,r)\} \subset \mathcal{F}.$$

- Let A be a non-negative measurable function in Ω

$$\int_{\mathcal{F}} \left(\int_{\Gamma_\alpha(Q)} A(X) dX \right) d\sigma(Q) \leq C_\alpha \int_{\mathcal{R}_\alpha(\mathcal{F})} A(X) \delta(X)^{n-1} dX.$$

- There is $\gamma \in (0,1)$ close to 1 such that

$$\int_{\mathcal{R}_\alpha(\mathcal{F}_\gamma^*)} A(Y) \delta(Y)^{n-1} dY \leq C_{\alpha,\gamma} \int_{\mathcal{F}} \left(\int_{\Gamma_\alpha(Q)} A(Y) dY \right) d\sigma(Q).$$

Square functions on CAD

For a measurable function f in Ω and $Q \in \partial\Omega$ define

$$A(f)(Q) = \left(\int_{\Gamma(Q)} f^2(X) \frac{dX}{\delta(X)^n} \right)^{1/2}$$

$$C(f)(Q) = \sup_{Q \in \Delta} \left(\frac{1}{\sigma(\Delta)} \int_{T(\Delta)} f(X)^2 \frac{dX}{\delta(X)} \right)^{1/2}$$

$$\mathcal{T}^p = \left\{ f \in L^2(\Omega) : A(f) \in L^p(\sigma) \right\}$$

$$\|f\|_{\mathcal{T}^p} = \|A(f)\|_{L^p(\sigma)}.$$

Duality of $\mathcal{T}^p$ spaces on CAD

- Let $1 < p < \infty$. The dual of $\mathcal{T}^q$ is $\mathcal{T}^p$ with $\frac{1}{p} + \frac{1}{q} = 1$, i.e. $G \in (\mathcal{T}^p)^*$ there exists $g \in \mathcal{T}^q$ such that for every $f \in \mathcal{T}^p$

$$G(f) = \int_\Omega f(X)g(X)\frac{dX}{\delta(X)}$$

  Moreover

$$||G|| \sim ||g||_{\mathcal{T}^q}.$$

- For $2 < p < \infty$

$$\|f\|_{\mathcal{T}^p} = \|A(f)\|_{L^p(\sigma)} \sim \|C(f)\|_{L^p(\sigma)}.$$

A lemma to illustrate the harmonic analysis results

Let $f, g \in L^2(\Omega)$, $f, g \geq 0$ then

$$\int_\Omega f(X)g(X)\frac{dX}{\delta(X)} \lesssim \int_{\partial\Omega} A(f)(Q)A(g)(Q)d\sigma(Q)$$

and

$$\int_\Omega f(X)g(X)\frac{dX}{\delta(X)} \lesssim \int_{\partial\Omega} C(f)(Q)A(g)(Q)d\sigma(Q)$$

Note for $\frac{1}{p} + \frac{1}{q} = 1$

$$\begin{aligned}\int_\Omega f(X)g(X)\frac{dX}{\delta(X)} &\lesssim \int_{\partial\Omega} A(f)(Q)A(g)(Q)d\sigma(Q) \\ &\lesssim \|f\|_{\mathcal{T}^p}\|g\|_{\mathcal{T}^q}\end{aligned}$$

A proof to illustrate the techniques

For $\tau > 0$ define the truncated cone

$$\Gamma^\tau(Q) = \{X \in \Omega : |X - Q| < 2\delta(X),\ \delta(X) \le \tau\},$$

$$A_\tau(f)(Q) := \left(\int_{\Gamma^\tau(Q)} f(X)^2 \frac{dX}{\delta(X)^n} \right)^{1/2}.$$

Note that $A_\tau(f)$ increases with τ and $A_\tau(f) = A(f)$ for $\tau > \operatorname{diam} \Omega$.
For Λ large let

$$\tau(Q) = \sup\{\tau > 0 : A_\tau(f)(Q) \le \Lambda C(f)(Q)\}$$

Claim 1: $\exists c_0 > 0$ such that for every $Q_0 \in \partial\Omega$ and $0 < r \le \operatorname{diam} \Omega$

$$\sigma(\{Q \in \Delta(Q_0, r) : \tau(Q) \ge r\}) \ge c_0 \sigma(\Delta(Q_0, r)).$$

Claim 2: For $H \geq 0$

$$\int_{\Omega} H(X)\delta(X)^{n-1} dX \lesssim \int_{\partial\Omega} \left\{ \int_{\Gamma^{\tau(Q)}(Q)} H(X) dX \right\} d\sigma(Q)$$

Proof:

- For $X \in \Omega$ there is $Q_X \in \partial\Omega$ such that $dist(X, \partial\Omega) = |X - Q|$.
- If $Q \in \Delta(Q_X, \delta(X))$ and $\delta(X) \leq \tau(Q)$ then $X \in \Gamma^{\tau(Q)}(Q)$ and

$$\chi_{\Gamma^{\tau(Q)}(Q)}(X) \geq \chi_{\Delta(Q_X, \delta(X)) \cap \{\tau(Q) \geq \delta(X)\}}(Q).$$

By Fubini, the remark above, claim 1 and Ahlfors regularity of σ:

$$\int_{\partial\Omega}\int_{\Gamma^{\tau(Q)}(Q)} H(X)dXd\sigma(Q) = \int_{\Omega}\int_{\partial\Omega} H(X)\chi_{\Gamma^{\tau(Q)}(Q)}(X)d\sigma(Q)dX$$

$$\begin{aligned}
&\geq \int_{\Omega}\int_{\partial\Omega} H(X)\chi_{\Delta(Q_X,\delta(X))\cap\{\tau(Q)\geq\delta(X)\}}(Q)d\sigma(Q)dX \\
&\geq \int_{\Omega} H(X)\sigma(\{Q\in\Delta(Q_X,\delta(X)):\tau(Q)\geq\delta(X)\})dX \\
&\geq c_0\int_{\Omega} H(X)\sigma(B(Q_X,\delta(X)))dX \\
&\gtrsim \int_{\Omega} H(X)\delta(X)^{n-1}dX.
\end{aligned}$$

Apply claim 2 to $H(X) = f(X)g(X)\delta(X)^{-n}$. Using Cauchy-Schwartz we have

$$\begin{aligned}\int_\Omega f(X)g(X)\frac{dX}{\delta(X)} &\lesssim \int_{\partial\Omega}\left(\int_{\Gamma^{\tau(Q)}(Q)} f(X)g(X)\delta(X)^{-n}dX\right)d\sigma(Q)\\ &\lesssim \int_{\partial\Omega}\left(\int_{\Gamma^{\tau(Q)}(Q)}\frac{f^2(X)}{\delta(X)^n}dX\right)^{\frac{1}{2}}\left(\int_{\Gamma^{\tau(Q)}(Q)}\frac{g^2(X)}{\delta(X)^n}dX\right)^{\frac{1}{2}}\\ &\lesssim \int_{\partial\Omega} A_{\tau(Q)}(f)(Q)A_{\tau(Q)}(g)(Q)d\sigma(Q).\end{aligned}$$

Since $A_{\tau(Q)}(f)(Q) \leq A(f)(Q)$ we have

$$\int_\Omega f(X)g(X)\frac{dX}{\delta(X)} \lesssim \int_{\partial\Omega} A(f)(Q)A(g)(Q)d\sigma(Q)$$

By definition of $\tau(Q)$,

$$A_{\tau(Q)}(f)(Q) \le \Lambda C(f)(Q)$$

hence

$$\begin{aligned} \int_\Omega f(X)g(X)\frac{dX}{\delta(X)} &\lesssim \int_{\partial\Omega} A_{\tau(Q)}(f)(Q)A_{\tau(Q)}(g)(Q)d\sigma(Q) \\ &\lesssim \int_{\partial\Omega} C(f)(Q)A(g)(Q)d\sigma(Q) \end{aligned}$$