Analysis on non-smooth domains

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Outline

- Lecture I : The Dirichlet Problem
  - Divergence form elliptic operators (Laplacian)
  - Domains (NTA & CAD)
  - Boundary regularity for harmonic functions
- Lecture II: Harmonic Analysis on CAD
  - Perturbation operators
  - $(D)_p$ problem: a PDE question becomes a Harmonic Analysis problem
  - Tent spaces on chord arc domains
- Lecture III: Boundary regularity results for perturbation operators
  - $(D)_p$ problem
  - Asymptotically optimal perturbation operators.

Divergence form elliptic operators

We consider operators of the form

$$Lu = \operatorname{div}(A(X)\nabla u)$$

where $A(X) = (a_{ij}(X))$ is symmetric measurable matrix and satisfies

$$\lambda|\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(X)\xi_i\xi_j \leq \Lambda|\xi|^2 \text{ for all } X \in \Omega \textit{ and } \xi \in \mathbb{R}^n.$$

- If $A = Id$, $L = \Delta$ the Laplacian.
- L is a variable coefficient version of the Laplacian.

Classical Dirichlet problem

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $f \in C(\partial\Omega)$ does there exist u satisfying

$$\left\{ \begin{array}{rcl} Lu = \operatorname{div}(A(X)\nabla u) & = & 0 \text{ in } \Omega \\ u & = & f \text{ on } \partial\Omega \text{ ?} \end{array} \right. \tag{1}$$

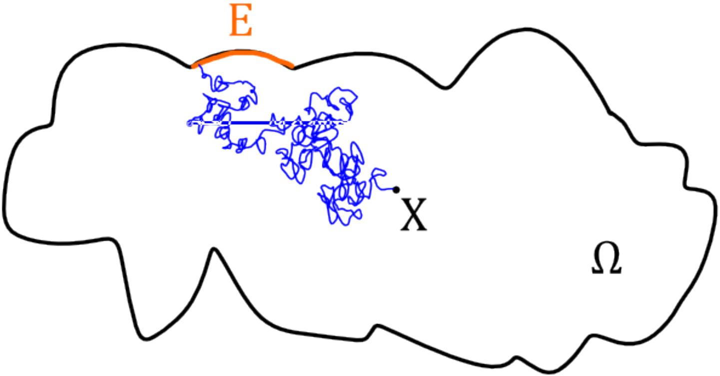
- If such u exists, how regular is it?

What is known about this question?

- The interior regularity is a classical result: u is Hölder continuous in Ω (De Giorgi-Nash-Moser).
- Additional regularity of A implies higher interior regularity of the solution.
- It is a question about boundary regularity.

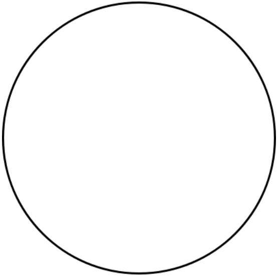
Elliptic measure

- Ω is regular for L, if for all $f \in C(\partial\Omega)$, $u_f = u \in C(\overline{\Omega})$.
- If Ω is regular the maximum principle and the Riesz Representation Theorem guarantee that there is a family probability measures $\{\omega_L^X\}_{X\in\Omega}$
$$u(X) = \int_{\partial\Omega} f(Q)d\omega_L^X(Q).$$
- ω_L is called the $L-$elliptic measure of Ω. If L is the Laplacian $\omega_L = \omega$ is the harmonic measure.
- The boundary regularity of u is determined by the regularity of ω_L.

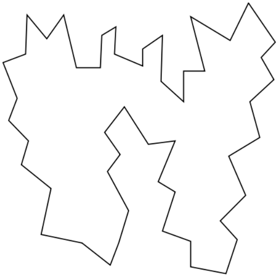


$\omega^X(E)$ denotes the probability that a Brownian motion starting at X will first hit the boundary at a point of $E \subset \partial\Omega$.

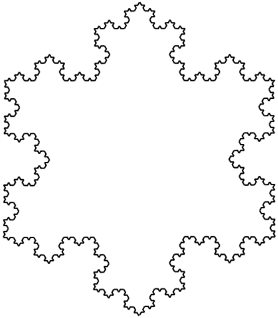
Examples of regular domains: non-tangentially accessible (NTA) domains



Smooth Domains



Lipschitz Domains



Quasispheres

(e.g. snowflake)

NTA domains - Jerison-Kenig

A domain Ω is non-tangentially accessible (NTA) if there exist $M > 2$ and $R > 0$ such that $\forall\, Q \in \partial\Omega$, $\forall\, r \in (0, R)$

1. Ω satisfies the corkscrew condition:

$$\exists A \in \Omega \qquad s.t. \qquad \frac{r}{M} \leq |A - Q|,\, d(A, \partial\Omega) \leq r$$

2. Ω^C satisfies the corkscrew condition.

3. Ω satisfies the Harnack Chain Condition;
if $\epsilon > 0$, $X_1, X_2 \in B(Q, r) \cap \Omega$ with $|X_1 - X_2| \leq 2^k \epsilon$ and $d(X_i, \partial\Omega) \geq \epsilon$ for $i = 1, 2$, there exists a chain of Mk balls $B_1, \ldots, B_{Mk}$ in Ω connecting $X_1 \in B_1$ to $X_2 \in B_{Mk}$ so that $diam\, B_j \sim d(B_j, \partial\Omega)$ and

$diam\, B_j \geq C^{-1} \min\{d(X_1, B_j), d(X_2, B_j)\}$ for $C > 1$.

Results on NTA domains - Jerison-Kenig

- NTA domains are regular
- ω_L is doubling, i.e. there exists a constant $C > 0$ such that for all $Q \in \partial\Omega$ and $0 < r < \operatorname{diam}\Omega$

$$\omega_L(B(Q, 2r)) \leq C\omega_L(B(Q, r)).$$

- The non-tangential limit of the solution of (**??**) at the boundary exists and coincides with f ω_L-a.e (i.e $u = f$ on $\partial\Omega$ ω_L-a.e.).
- If f is Lipschitz, u is Hölder continuous in $\overline{\Omega}$.

Chord Arc Domains (CAD)

A chord arc domain (CAD) is an NTA domain whose surface measure at the boundary σ is Ahlfors regular, i.e. there exists $C > 1$ such that for all $Q \in \partial\Omega$ and $r \in (0, \operatorname{diam}\Omega)$

$$C^{-1} r^{n-1} \leq \sigma(B(Q, r)) \leq C r^{n-1}.$$

Here $\sigma = \mathcal{H}^{n-1} \llcorner \partial\Omega$, where $\mathcal{H}^{n-1}$ denotes the $(n-1)$-dimensional Hausdorff measure.

Examples:

- Lipschitz domains.
- Domains which locally can be seen as the area above the graph of a function with gradient in BMO.

CAD

Remarks:

- Locally the boundary of a CAD is not necessarily the graph of a function.
- CADs may appear as Hausdorff limits of smooth domains.
- CADs are sets of locally finite perimeter.

Boundary regularity result:

- Semmes & David-Jerison: If $L = \Delta$, $\omega \in A_\infty(\sigma)$ i.e. $\omega \ll \sigma$ in a quantitative way.

Chord arc domains with small constant

- A bounded domain $\Omega \subset \mathbb{R}^n$ is a δ-CAD, if Ω is a CAD, Ω is δ-Reifenberg flat and there exists $R > 0$ such that

$$\|\nu\|_*(R) = \sup_{0<r<R} \sup_{Q\in\partial\Omega} \left(\frac{1}{\sigma(B(Q,r))} \int_{B(Q,r)} |\nu - \nu_{Q,r}|^2 \, d\sigma \right)^{1/2} < \delta$$

where

$$\nu_{Q,r} = \frac{1}{\sigma(B(Q,r))} \int_{B(Q,r)} \nu \, d\sigma$$

- Examples: Domains which locally can be seen as the area above the graph of a Lipschitz function with constant comparable to δ or a function whose gradient has BMO norm comparable to δ.

Chord arc domains with vanishing constant

- A bounded domain $\Omega \subset \mathbb{R}^n$ is a CAD with vanishing constant if Ω is a δ-CAD for some δ small and $\nu \in VMO(\sigma)$, i.e.

$$\limsup_{s \to 0} \|\nu\|_*(s) = 0.$$

- Examples:
  - C^1 domains
  - Domains which locally can be seen as the area above the graph of a function whose gradient is in VMO

Harmonic measure on Lipschitz domains vs CAD

Recall $A_\infty(\sigma) = \cup_{q>1} B_q(\sigma)$ and $\omega \in B_q(\sigma)$ for $q > 1$ if the Poisson kernel $k = \frac{d\omega}{d\sigma}$ satisfies a reverse Hölder inequality, i.e.

$$\left(\frac{1}{\sigma(\Delta)} \int_\Delta k^q \, d\sigma\right)^{\frac{1}{q}} \leq C \frac{1}{\sigma(\Delta)} \int_\Delta k \, d\sigma$$

for all $\Delta = B(Q, r) \cap \partial\Omega$ with $Q \in \partial\Omega$ and $r \in (0, \, diam\Omega)$.

- Dahlberg: If Ω is Lipschitz then $\omega \in A_\infty(\sigma)$ and $\omega \in B_2(\sigma)$
- David-Jerison & Semmes: If Ω is a CAD there exists $q \in (1, \infty)$ such that $\omega \in B_q(\sigma)$. Given $q > 1$ there exists a CAD, Ω, such that $\omega \notin B_q(\sigma)$.
- Kenig-Toro: Given $q > 1$ there exists $\delta > 0$ such that if Ω is a δ-CAD then $\omega \in B_q(\sigma)$. (Key tool: Semmes local decomposition of chord arc surfaces with small constant)

Poisson kernel on C^1 domains vs CAD with vanishing constant

- Jerison-Kenig: If Ω is a C^1 domain then $\log k \in VMO(\sigma)$.
- Kenig-Toro: If Ω is a chord arc domain with vanishing constant then $\log k \in VMO(\sigma)$.
- Fabes-Jodeit-Rivière: If Ω is a C^1 domain then the double layer potential is a compact operator from $L^p(\sigma)$ into $L^p(\sigma)$ for any $1 < p < \infty$.
- Hofmann-Mitrea-Taylor: If Ω is a chord arc domain with vanishing constant then the double layer potential is a compact operator from $L^p(\sigma)$ into $L^p(\sigma)$ for any $1 < p < \infty$.

What does the regularity of the Poisson kernel tell us about the boundary regularity of the solutions to the Dirichlet problem with data in L^p?

- Given $f \in L^p(\sigma) \cap C(\partial\Omega)$ with $p > 1$ does there exist u such that

$$\begin{cases} Lu = \operatorname{div}(A(X)\nabla u) &= 0 \text{ in } \Omega \\ u &= f \text{ on } \partial\Omega \end{cases}$$

with

$$\|N(u)\|_{L^p(\sigma)} \leq C\|f\|_{L^p(\sigma)} \quad \text{where} \quad N(u) = \sup_{X \in \Gamma(Q)} |u(X)| \, ?$$

$$\Gamma(Q) = \{X \in \Omega : |X - Q| \leq 2\delta(X)\} \quad \text{and} \quad \delta(X) = \operatorname{dist}(X, \partial\Omega)$$

$(D)_p$ problem & Weights

- If the $(D)_p$ problem holds then for $\sigma - a.e.\ Q \in \partial\Omega$, $\lim_{X\to Q, X\in\Gamma(Q)} u(X) = f(Q)$.
- The $(D)_p$ holds for L if and only if $\omega_L \in B_q(\sigma)$ with $\frac{1}{p} + \frac{1}{q} = 1$.
- $\omega_L \in B_q(\sigma)$ if $k_L = \frac{d\omega_L}{d\sigma}$ satisfies

$$\left(\frac{1}{\sigma(\Delta)}\int_\Delta k_L^q\, d\sigma\right)^{\frac{1}{q}} \leq C\frac{1}{\sigma(\Delta)}\int_\Delta k_L\, d\sigma$$

  for all $\Delta = B(Q, r) \cap \partial\Omega$ with $Q \in \partial\Omega$ and $r \in (0, diam\Omega)$.
- Note if $\omega_L \in B_q(\sigma)$ then $\omega_L \in B_{q'}(\sigma)$ for $q' < q$.

Summary of results for the Laplacian

- Dahlberg: If Ω is a Lipschitz domain then $\omega \in B_2(\sigma)$, and $(D)_p$ problem for the Laplacian holds for all $p \geq 2$.

- Toro-Kenig: Given $p_0 > 1$ there exists $\delta > 0$ such that if Ω is a δ-CAD then $(D)_p$ problem for the Laplacian holds for all $p \geq p_0$.

- Toro-Kenig & Hofmann-Mitrea-Taylor : If Ω is a chord arc domain with vanishing constant then $(D)_p$ problem for the Laplacian holds for all $p > 1$.