Quasi-greedy bases and Lebesgue-type inequalities

Vladimir Temlyakov

University of South Carolina

Steklov Institute of Mathematics

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1. Greedy approximation

Let a Banach space X with a normalized basis $\Psi = \{\psi_k\}_{k=1}^{\infty}$, $\|\psi_k\| = 1$, $k = 1, 2, \dots$, be given. We consider the following greedy algorithm that we call the Thresholding Greedy Algorithm (TGA). For a given element $f \in X$ we consider the expansion

$$f = \sum_{k=1}^{\infty} c_k(f)\psi_k.$$

Let an element $f \in X$ be given. We call a permutation ρ, $\rho(j) = k_j$, $j = 1, 2, \dots$, of the positive integers decreasing and write $\rho \in D(f)$ if

$$|c_{k_1}(f)| \geq |c_{k_2}(f)| \geq \dots \quad .$$

Greedy approximant

In the case of strict inequalities here $D(f)$ consists of only one permutation. We define the m-th greedy approximant of f with regard to the basis Ψ corresponding to a permutation $\rho \in D(f)$ by the formula

$$G_m(f, \Psi) := G_m(f, \Psi, \rho) := \sum_{j=1}^{m} c_{k_j}(f)\psi_{k_j}.$$

Greedy versus best

In order to understand the efficiency of this algorithm we compare its accuracy with the best possible

$$\sigma_m(f, \Psi) := \sigma_m(f, \Psi)_X := \inf_{c_k, \Lambda; |\Lambda| = m} \|f - \sum_{k \in \Lambda} c_k \psi_k\|_X,$$

when an approximant is a linear combination of m terms from Ψ. The best we can achieve with the algorithm G_m is

$$\|f - G_m(f, \Psi, \rho)\| = \sigma_m(f, \Psi),$$

or a little weaker: for all elements $f \in X$

$$\|f - G_m(f, \Psi, \rho)\| \leq G\sigma_m(f, \Psi) \tag{1.1}$$

with a constant $G = C(X, \Psi)$ independent of f and m.

Definition of greedy basis

Definition 1.1. We call a basis Ψ greedy basis if for every $f \in X$ there exists a permutation $\rho \in D(f)$ such that (1.1) holds.

The following proposition has been proved in [Konyagin, T., 1999].

Proposition 1.1. If Ψ is a greedy basis then (1.1) holds for any permutation $\rho \in D(f)$.

Trigonometric system

We proved in [T., 1998] the following results.

Theorem 1.1. For each $f \in L_p(\mathbb{T}^d)$ we have

$$\|f - G_m(f,\mathcal{T})\|_p \le (1 + 3m^{h(p)})\sigma_m(f,\mathcal{T})_p, \quad 1 \le p \le \infty,$$

where $h(p) := |1/2 - 1/p|$.

Remark 1.1. There is a positive absolute constant C such that for each m and $1 \le p \le \infty$ there exists a function $f \neq 0$ with the property

$$\|G_m(f,\mathcal{T})\|_p \ge C m^{h(p)} \|f\|_p.$$

Haar system

Denote $\mathcal{H}_p := \{H_k^p\}_{k=1}^{\infty}$ the Haar basis on $[0,1)$ normalized in $L_p(0,1)$: $H_1^p = 1$ on $[0,1)$ and for $k = 2^n + l$, $l = 1, 2, \ldots, 2^n$, $n = 0, 1, \ldots$

$$H_k^p = \begin{cases} 2^{n/p}, & x \in [(2l-2)2^{-n-1}, (2l-1)2^{-n-1}) \\ -2^{n/p}, & x \in [(2l-1)2^{-n-1}, 2l2^{-n-1}) \\ 0, & \text{otherwise.} \end{cases}$$

Haar basis is a greedy basis

The following theorem establishes the existence of greedy bases for $L_p(0,1)$, $1 < p < \infty$.

Theorem 1.2 (T., 1998). Let $1 < p < \infty$ and a basis Ψ be L_p-equivalent to the Haar basis $\mathcal{H}_p$. Then for any $f \in L_p(0,1)$ and any $\rho \in D(f)$ we have

$$\|f - G_m(f,\Psi,\rho)\|_{L_p} \leq C(p,\Psi)\sigma_m(f,\Psi)_{L_p}$$

with a constant $C(p,\Psi)$ independent of f, ρ, and m.

Theorem 1.2 establishes that each basis Ψ which is L_p-equivalent to the univariate Haar basis $\mathcal{H}_p$ is a greedy basis for $L_p(0,1)$, $1 < p < \infty$. We note that in the case of Hilbert space each orthonormal basis is a greedy basis with a constant $G = 1$ (see (1.1)).

L_p-equivalence

In this theorem we use the following definition of the L_p-equivalence. We say that $\Psi = \{\psi_k\}_{k=1}^{\infty}$ is L_p-equivalent to $\mathcal{H}_p = \{H_k^p\}_{k=1}^{\infty}$ if for any finite set K and any coefficients c_k, $k \in K$, we have

$$C_1(p,\Psi)\|\sum_{k\in K} c_k H_k^p\|_{L_p} \leq \|\sum_{k\in K} c_k \psi_k\|_{L_p} \leq C_2(p,\Psi)\|\sum_{k\in K} c_k H_k^p\|_{L_p}$$

with two positive constants $C_1(p,\Psi), C_2(p,\Psi)$ which may depend on p and Ψ. For sufficient conditions on Ψ to be L_p-equivalent to $\mathcal{H}_p$ see [Frazier, Jawerth, 1990] and [DeVore, Konyagin, T., 1998].

Unconditional basis

We give the definitions of unconditional and democratic bases.

Definition 1.2. A basis $\Psi = \{\psi_k\}_{k=1}^{\infty}$ of a Banach space X is said to be unconditional if for every choice of signs $\theta = \{\theta_k\}_{k=1}^{\infty}$, $\theta_k = 1$ or -1, $k = 1, 2, \dots$, the linear operator M_θ defined by

$$M_\theta(\sum_{k=1}^{\infty} a_k \psi_k) = \sum_{k=1}^{\infty} a_k \theta_k \psi_k$$

is a bounded operator from X into X.

Democratic basis

Definition 1.3. We say that a basis $\Psi = \{\psi_k\}_{k=1}^{\infty}$ is a democratic basis if for any two finite sets of indices P and Q with the same cardinality $|P| = |Q|$ we have

$$\|\sum_{k\in P}\psi_k\| \le D\|\sum_{k\in Q}\psi_k\|$$

with a constant $D := D(X, \Psi)$ independent of P and Q.

Characterization

We proved in [Konyagin, T., 1999] the following theorem.
Theorem 1.3. A basis is greedy if and only if it is unconditional and democratic.
This theorem gives a characterization of greedy bases.
Further investigations ([T., 1998], [Cohen, DeVore, Hochmuth, 2000], [Kerkyacharian, Picard, 2004], [Gribonval, Nielsen, 2001], [Kamont, T., 2004]) showed that the concept of greedy bases is very useful in direct and inverse theorems of nonlinear approximation and also in applications in statistics.

2. Almost greedy bases

Let us discuss a question of weakening the property of a basis of being a greedy basis. We begin with a concept of quasi-greedy basis introduced in [Konyagin, T., 1999].

Definition 2.1. We call a basis Ψ quasi-greedy basis if for every $f \in X$ and every permutation $\rho \in D(f)$ we have

$$\|G_m(f, \Psi, \rho)\|_X \leq C\|f\|_X \tag{2.1}$$

with a constant C independent of f, m, and ρ.

P. Wojtaszczyk, 2000, proved the following theorem.

Theorem 2.1. A basis Ψ is quasi-greedy if and only if for any $f \in X$ and any $\rho \in D(f)$ we have

$$\|f - G_m(f, \Psi, \rho)\| \to 0 \quad \text{as} \quad m \to \infty. \tag{2.2}$$

Best expansional approximation

We proceed to a concept of almost greedy basis. This concept was introduced and studied in [Dilworth, Kalton, Kutzarova, T., 2003]. Let

$$f = \sum_{k=1}^{\infty} c_k(f)\psi_k.$$

We define the following expansional best m-term approximation of f

$$\tilde{\sigma}_m(f) := \tilde{\sigma}_m(f, \Psi) := \inf_{\Lambda, |\Lambda|=m} \|f - \sum_{k\in\Lambda} c_k(f)\psi_k\|.$$

It is clear that $\sigma_m(f, \Psi) \leq \tilde{\sigma}_m(f, \Psi)$.

Definition of almost greedy basis

It is also clear that for an unconditional basis Ψ we have

$$\tilde{\sigma}_m(f, \Psi) \leq C\sigma_m(f, \Psi).$$

Definition 2.2. We call a basis Ψ almost greedy basis if for every $f \in X$ there exists a permutation $\rho \in D(f)$ such that

$$\|f - G_m(f, \Psi, \rho)\|_X \leq C\tilde{\sigma}_m(f, \Psi)_X \quad (2.3)$$

holds with a constant independent of f, m.

The following proposition follows from [Dilworth, Kalton, Kutzarova, T., 2003].

Proposition 2.1. If Ψ is an almost greedy basis then (2.3) holds for any permutation $\rho \in D(f)$.

Characterization

The following characterization of almost greedy bases was obtained in [Dilworth, Kalton, Kutzarova, T., 2003].

Theorem 2.1. Suppose Ψ is a basis of a Banach space. The following are equivalent:

A. Ψ is almost greedy.

B. Ψ is quasi-greedy and democratic.

C. For any $\lambda > 1$ there is a constant $C = C_\lambda$ such that

$$\|f - G_{[\lambda m]}(f, \Psi)\| \leq C_\lambda \sigma_m(f, \Psi).$$

Relations

We have discussed the following bases.

1. Unconditional;
2. Democratic;
3. Quasi-greedy;
4. Greedy;
5. Almost greedy.

We have formulated the following relations.

Unconditional + Democratic = Greedy

Quasi-greedy + Democratic = Almost greedy

We formulate some relations between the above bases.

Unconditional implies Quasi-greedy

Quasi-greedy does not imply Unconditional

Unconditional does not imply Democratic

Democratic does not imply Unconditional

Greedy implies Almost greedy

Almost greedy does not imply Greedy

These properties follow from [Konyagin, T., 1999].

3. The Lebesgue inequality

A. Lebesgue proved the following inequality: for any 2π-periodic continuous function f one has

$$\|f - S_n(f)\|_\infty \leq (4 + \frac{4}{\pi^2} \ln n) E_n(f)_\infty,$$

where $S_n(f)$ is the nth partial sum of the Fourier series of f and $E_n(f)_\infty$ is the error of the best approximation of f by the trigonometric polynomials of order n in the uniform norm $\| \cdot \|_\infty$.

The first form of Lebesgue inequality

There are two natural ways of adapting (1.1) to the case of nongreedy basis. In the first way (see [T., 1998], [Wojtaszczyk, 2000], [Oswald, 2000]) we write (1.1) in the form

$$\|f - G_m(f, \Psi)\| \leq C(m, \Psi)\sigma_m(f, \Psi)$$

and look for the best (in the sense of order) constant $C(m, \Psi)$ in the above Lebesgue type inequality.

Fundamental functions

For a basis Ψ we define the fundamental function $\varphi(m)$ and the functions $\varphi^s(n)$ and $\phi(n)$:

$$\varphi^s(n) := \sup_{|A|=n} \|\sum_{k\in A} \psi_k\|.$$

$$\varphi(m) := \sup_{n\leq m} \varphi^s(n);$$

$$\phi(n) := \inf_{|A|=n} \|\sum_{k\in A} \psi_k\|.$$

Characteristics of a basis

Define

$$\mu(m) := \sup_{n \le m} \frac{\varphi^s(n)}{\phi(n)}.$$

The characteristics $\varphi^s(n)$, $\phi(n)$ and $\mu(m)$ were used in the first papers on greedy approximation with respect to bases. They were used in [T, 1998] for the multivariate Haar basis $\mathcal{H}^d := \mathcal{H} \times \cdots \times \mathcal{H}$, then they were used in [Wojtaszczyk, 2000], [Kamont and T., 2004], Garrigos, Hernandez, Natividade, 2011 and in other papers.

Lebesgue type inequality I

The following result has been proved in [Kamont and T., 2004].

Theorem 3.1. Let Ψ be a normalized unconditional basis for X. Then we have

$$\|f - G_m(f, \Psi)\| \leq C(\Psi)\mu(m)\sigma_m(f, \Psi).$$

In Theorem 3.1 we compare efficiency of $G_m(\cdot, \Psi)$ with $\sigma_m(\cdot, \Psi)$. It is known in approximation theory that sometimes it is convenient to compare efficiency of an approximating operator which is characterized by m parameters with best possible approximation corresponding to smaller number of parameters $n \leq m$. We use this idea in approximation by the TGA.

The second form of Lebesgue inequality

Let us discuss a setting (see [Kamont and T., 2004]) when we write (1.1) in the form

$$\|f - G_{v_m}(f, \Psi)\| \leq C(\Psi)\sigma_m(f, \Psi) \tag{3.1}$$

and look for the best (in the sense of order) sequence $\{v_m\}$ that is determined by the weakness sequence τ and the basis Ψ. Inequalities of the type (3.1) can also be called de la Vallée Poussin inequalities.

Lebesgue type inequality II

Assume that $\phi(N) \to \infty$ as $N \to \infty$ and denote v_m the smallest N satisfying

$$\phi(N) \geq 2\varphi(m).$$

There is the following Lebesgue type inequality in this case ([Kamont and T., 2004]).

Theorem 3.2. For any normalized unconditional basis Ψ we have

$$\|f - G_{v_m}(f, \Psi)\| \leq C(\Psi)\sigma_m(f, \Psi).$$

Almost greedy basis

It is interesting to compare this result with results from [Dilworth, Kalton, Kutzarova, and T., 2003]. It has been established in the above mentioned paper that the inequalities

$$\|f - G_{[\lambda m]}(f, \Psi)\| \leq C(\Psi, \lambda)\sigma_m(f, \Psi) \tag{3.2}$$

with fixed $\lambda > 1$ are characteristic for a class of almost greedy bases. Each greedy basis is an almost greedy basis. There is an example (see [Konyagin and T., 1999]) of almost greedy basis that is not a greedy basis. This means that $\lambda > 1$ needed for (3.2) can not be replaced by $\lambda \geq 1$.

4. Quasi-greedy bases

We begin with some Lebesgue-type inequalities for greedy approximation with respect to a quasi-greedy basis from [T., Yang and Ye, 2011]. Here is an analog of Theorem 3.1.

Theorem 4.1. Let Ψ be a quasi-greedy basis of X satisfying the following assumption: There exists an increasing function $v(m) := v(m,\Psi)$ such that for any two sets of indices A and B, $|A| = |B| = m$ we have

$$\|\sum_{k\in A}\psi_k\| \le v(m)\|\sum_{k\in B}\psi_k\|.$$

Then for each $f \in X$

$$\|f - G_m(f)\| \le C(\Psi, X)v(m)\tilde{\sigma}_m(f).$$

Lebesgue-type inequality in L_p

The following theorem is an analog of Theorem 1.1.
Theorem 4.2. Let $1 < p < \infty$, $p \neq 2$, and let Ψ be a quasi-greedy basis of the L_p space. Then for each $f \in L_p$ we have

$$\|f - G_m(f,\Psi)\|_{L_p} \leq C(p,\Psi) m^{|1/2-1/p|} \sigma_m(f,\Psi)_{L_p}. \tag{4.1}$$

This theorem is based on the following Theorem 4.3 from [T., Yang and Ye, 2010] that is interesting by itself. We note that in the case $p = 2$ Theorem 4.3 was proved in [P. Wojtaszczyk, 2000]. We will use the notation

$$a_n(f) := |c_{k_n}(f)|$$

for the decreasing rearrangement of the coefficients of f.

Bounds for L_p norm

Theorem 4.3. Let $\Psi = \{\psi_k\}_{k=1}^{\infty}$ be a quasi-greedy basis of the L_p space, $1 < p < \infty$. Then for each $f \in X$ we have for $2 \leq p < \infty$

$$C_1(p) \sup_n n^{1/p} a_n(f) \leq \|f\|_p \leq C_2(p) \sum_{n=1}^{\infty} n^{-1/2} a_n(f)$$

and for $1 < p \leq 2$

$$C_3(p) \sup_n n^{1/2} a_n(f) \leq \|f\|_p \leq C_4(p) \sum_{n=1}^{\infty} n^{1/p-1} a_n(f).$$

$p = 2$

The following result is from [T., Yang and Ye, 2010].
Theorem 4.4. Let Ψ be a normalized quasi-greedy basis of a Hilbert space H. Then, for any $f \in H$ and $\lambda > 1$

$$\|f - G_{\lambda m}(f, \Psi)\| \leq C(\lambda)\sigma_m(f, \Psi).$$

We note that if in Theorem 4.4 $G_{\lambda m}$ can be replaced by G_m then Ψ is a greedy basis. It is known ([P. Wojtaszczyk, 2000]) that for a separable, infinite dimensional Hilbert space H there exists a quasi-greedy basis that is not an unconditional basis. Therefore, this basis is not a greedy basis. Thus, one cannot replace the restriction $\lambda > 1$ by $\lambda \geq 1$ in Theorem 4.4.

$\lambda = 1$

It is mentioned in [P. Wojtaszczyk, 2000] that in the case $\lambda = 1$ one has the following inequality

$$\|f - G_m(f, \Psi)\| \le C(\log m)\sigma_m(f, \Psi).$$

We do not know if the above inequality is sharp in the sense that an extra factor $\log m$ cannot be replaced by a slower growing factor.

5. Recent results

In [Dilworth, Soto-Baho and T., 2012] we prove that if Ψ is both quasi-greedy and democratic then for any $f \in X$

$$\|f - G_m(f,\Psi)\|_X \le C \ln(m+1)\sigma_m(f,\Psi)_X. \tag{5.1}$$

We note that quasi-greedy and democratic are exactly almost greedy bases. Using (5.1) we obtain the Lebesgue-type inequality for a uniformly bounded quasi-greedy basis of L_p, $1 < p < \infty$:

$$\|f - G_m(f,\Psi)\|_p \le C(p) \ln(m+1)\sigma_m(f,\Psi)_p. \tag{5.2}$$

Here $\sigma_m(f,\Psi)_p := \sigma_m(f,\Psi)_{L_p}$. Comparing (5.2) with (4.1) we see that an extra assumption of uniform boundedness of the basis improves the Lebesgue-type inequalities dramatically.

Expansional versus best

We note that (5.1) is an easy corollary of the following inequality

$$\tilde{\sigma}_m(f,\Psi) \le C \ln(m+1)\sigma_m(f,\Psi) \tag{5.3}$$

that holds for any quasi-greedy basis Ψ. The question if (5.3) is true was formulated by [Hernandez, 2011].

The (5.3) was proved independently in [Dilworth, Soto-Bajo, and T., 2012] and [Garrigos, Hernandez, and Oikhberg, 2012].

Uniformly bounded orthonormal

In [Dilworth, Soto-Bajo, and T., 2012], making our assumptions on the basis even stronger, we improve (5.2) to the following inequality

$$\|f - G_m(f, \Psi)\|_p \leq C(p)(\ln(m+1))^{1/2}\sigma_m(f, \Psi)_p, \qquad (5.4)$$

under assumption that Ψ is a uniformly bounded orthonormal quasi-greedy basis of L_p, $2 \leq p < \infty$.

Uniformly bounded, different q and p

In [Dilworth, Soto-Bajo, and T., 2012] we impose assumptions on the basis in the L_q space and obtain inequalities in the L_p space:

$$\|f - G_m(f,\Psi)\|_p \leq C(p,q)m^{(1-q/p)/2}\ln(m+1)\sigma_m(f,\Psi)_p \qquad (5.5)$$

under assumption that Ψ is a uniformly bounded quasi-greedy basis of L_q, $1 < q < \infty$, and $q \leq p \leq \infty$. We note that in the case $p = q$ inequality (5.5) turns into (5.2).