Cauchy non-integral formulas

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June 2012

El Escorial: from 2008 to 2012

- Alternative title: "semigroup methods for elliptic systems"
  (Evolution parameter t = variable transversal to the boundary.)
- A survey of the understanding 2008 of these techniques and applications to equations with t-independent coefficients:
  P. Auscher, A. Axelsson, A. McIntosh: On a quadratic estimate related to the Kato conjecture and boundary value problems. *Contemporary Mathematics*.
- Our method for solving the PDEs uses operator theory for non-selfadjoint differential operators with non-smooth coefficients. The techniques from the solution of the Kato square root problem yield the crucial estimates for these operators.
- A strength of these techniques is that they apply to quite general elliptic systems.
  In this talk: divergence form equations.

Problem formulation

Space $\mathbf{R}^{1+n}$, where $n \geq 1$
Upper half space: $\mathbf{R}^{1+n}_+ := \{(t,x) \ ; \ t > 0, x \in \mathbf{R}^n\}$
Boundary: $\mathbf{R}^n := \{(0,x) \ ; \ x \in \mathbf{R}^n\}$

Question (Generalized Cauchy formulas)

For solutions to a given divergence form equation $\mathrm{div} A(t,x) \nabla u(t,x) = 0$ in $\mathbf{R}^{1+n}_+$, is there Cauchy type formula

$$\nabla u|_{\mathbf{R}^n} \mapsto \nabla u|_{\mathbf{R}^{1+n}_+}$$

for the gradient vector field? We ask this for the trace spaces

1. $\nabla u|_{\mathbf{R}^n} \in L_2(\mathbf{R}^n)$ $\Leftrightarrow \partial_{\nu_A} u \in L_2(\mathbf{R}^n)$ and $\nabla_{\|} u \in L_2(\mathbf{R}^n)$
2. $\nabla u|_{\mathbf{R}^n} \in H^{-1}(\mathbf{R}^n)$ $\Leftrightarrow u \in L_2(\mathbf{R}^n)$ and conjugates $\tilde{u} \in L_2(\mathbf{R}^n)$.

Such Cauchy formulas will provide a way to construct solutions to $\mathrm{div} A(t,x) \nabla u(t,x) = 0$ in $\mathbf{R}^{1+n}_+$ and to prove estimates of such.

A trivial example: $n = 1$, $A = I$

- For any harmonic function u, we have the Cauchy formula

$$\nabla u(z) = \frac{i}{2\pi} \int_{\mathbf{R}} \frac{\nabla u(y)}{y - \overline{z}} dy, \qquad z \in \mathbf{R}^2_+,$$

  since ∇u is an anti-analytic function in the half plane.

- Given $\phi : \mathbf{R} \to \mathbf{R}^2 = \mathbf{C}$, we can construct a vector field in $\mathbf{R}^2_+$

$$f(z) := \frac{i}{2\pi} \int_{\mathbf{R}} \frac{\phi(y)}{y - \overline{z}} dy, \qquad z \in \mathbf{R}^2_+,$$

  which is the gradient $f = \nabla u$ of a harmonic function u.

Assumptions on coefficients

- We consider general bounded coefficients $A \in L_\infty(\mathbf{R}^{1+n}_+; \mathcal{L}(\mathbf{C}^{1+n}))$ which are accretive in the sense that for all $v \in \mathbf{C}^{1+n}$ and almost all $(t,x) \in \mathbf{R}^{1+n}_+$
$$\mathrm{Re}(A(t,x)v, v) \geq \kappa > 0.$$
(For general systems, a weaker Gårding type inequality suffices.)
- Besides t-independent coefficients $A(t,x) = A(0,x)$, we allow coefficents with boundary continuity $\lim_{t\to 0} A(t,x) = A(0,x)$ in a certain Dahlberg/Carleson sense (to be specified later).
- This covers domains D, including non-graph domains, which can be bilipschitz parametrized by the half space, with appropriate Carleson control of the second derivatives: $\mathrm{div} A\nabla u = 0$ in D is equivalent to $\mathrm{div} A_\rho \nabla u_\rho$ in $\mathbf{R}^{1+n}_+$ under pullback $u \mapsto u_\rho$.

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Papers that this talk is based on

1. P. Auscher, A. Axelsson:
   Weighted maximal regularity estimates and solvability of non-smooth elliptic systems I. *Inventiones Mathematicae.*
   (Cauchy formulas on domains bilipschitz equivalent to the half space and applications to BVPs.)
2. P. Auscher, A. Rosén:
   Weighted maximal regularity estimates and solvability of non-smooth elliptic systems II. *Analysis & PDE.*
   (Cauchy formulas on domains bilipschitz equivalent to the unit ball and applications to BVPs.)
3. T. Hytönen, A. Rosén:
   On the Carleson duality. *Arkiv för Matematik.*
   (Duality results for a new scale of tent-type space.)
4. A. Rosén:
   Layer potentials beyond singular integral operators. *Preprint.*
   (Functional calculus vs. layer potentials.)

Generalized CR-systems for conormal gradients

- For solutions $\mathrm{div} A\nabla u = 0$ the natural transversal derivative is the conormal derivative
$$\partial_{\nu_A} u := (A\nabla u)_\perp = a\partial_t u + b\nabla_\| u, \quad \text{if } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$
- The *conormal gradient* of u is $f = \begin{bmatrix} f_\perp \\ f_\| \end{bmatrix} = \nabla_A u := \begin{bmatrix} \partial_{\nu_A} u \\ \nabla_\| u \end{bmatrix}$.
- View $f(t,x)$ as $(0,\infty) \ni t \mapsto f_t = f(t,\cdot) \in L_2(\mathbf{R}^n; \mathbf{C}^{1+n})$.

Proposition (div-form elliptic = vector valued ODE)

$$\mathrm{div} A\nabla u = 0 \Leftrightarrow \begin{cases} \partial_t f + DBf = 0, \\ \mathrm{curl}_\| f_\| = 0 \Leftrightarrow \forall t : f_t \in \overline{R(D)}, \end{cases}$$

$$D := \begin{bmatrix} 0 & \mathrm{div}_\| \\ -\nabla_\| & 0 \end{bmatrix}, \qquad B := \begin{bmatrix} a^{-1} & -a^{-1}b \\ ca^{-1} & d - ca^{-1}b \end{bmatrix}$$

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The infinitesimal generator DB_0

$B = \begin{bmatrix} a^{-1} & -a^{-1}b \\ ca^{-1} & d - ca^{-1}b \end{bmatrix}$ accretive $\Leftrightarrow A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ accretive. In particular

$$(Bv, v) \in S_\omega := \{\lambda \in \mathbf{C} \ ; \ |\arg \lambda| \leq \omega\}, \quad \text{for some } \omega < \pi/2.$$

Definition (tangential operators in the boundary space $L_2(\mathbf{R}^n; \mathbf{C}^{1+n})$)

$D = \begin{bmatrix} 0 & \mathrm{div}_\parallel \\ -\nabla_\parallel & 0 \end{bmatrix}$ and $B_0 : f(x) \mapsto B(0, x)f(x)$.

- The operator DB_0 is closed, densely defined (with infinite-dimensional nullspace if $n \geq 2$) and spectrum contained in the bisector $(-S_\omega) \cup S_\omega$.
- DB_0 induces a topological (but in general non-orthogonal) splitting of L_2 into spectral subspaces

$$L_2 = E_0^- L_2 \oplus E_0^0 L_2 \oplus E_0^+ L_2 \quad (= \mathsf{N}(DB_0) \oplus \overline{\mathsf{R}(DB_0)})$$

associated with $-S_\omega \setminus \{0\}$, $\{0\}$ and $S_\omega \setminus \{0\}$.

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H^∞ functional calculus for DB_0

- If b is a bounded holomorphic function on (an open sector slightly larger than) S_ω, then $b(DB_0)$ is bounded on $E_0^+ L_2$ with

$$\|b(DB_0)\|_{E_0^+ L_2 \to E_0^+ L_2} \leq C \sup_\lambda |b(\lambda)|.$$

- If $b(0) \in \mathbf{C}$ is defined, then define the operator $b(DB_0) := b(0)I$ on $E_0^0 L_2$.
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Operators for the ODE $\partial_t f + DB_0 f = 0$

- $e^{-tDB_0} = e^{-t(\cdot)}(DB_0)$ is bounded on $\begin{cases} E_0^+ L_2, & t > 0, \\ E_0^- L_2, & t < 0. \end{cases}$
- $E_0^+ = \chi^+(DB_0)$ is obtained from $\chi^+(\lambda) = \begin{cases} 1, & \mathrm{Re}\,\lambda > 0, \\ 0, & \mathrm{Re}\,\lambda \le 0. \end{cases}$
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Hardy type subspaces:

$$f_0 \in E_0^+ L_2 \Leftrightarrow f_0 = \lim_{t \to 0^+} f_t \quad \text{for some solution} \quad \partial_t f + DB_0 f = 0, t > 0.$$

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$1/2$ Cauchy integral = layer potential operators

In case of t-independent coefficients $A(t,x) = A(0,x)$, the Cauchy formula for $f(t,x) = \nabla_A u(t,x)$ is

$$f(t,x) = e^{-tDB_0} E_0^+ f_0(x), \qquad t > 0.$$

The following new result shows that at least $1/2$ of this operator is a classical singular integral.

Theorem (Rosén)

Consider $\mathrm{div} A \nabla u = 0$, where A are scalar real t-independent coefficients. Then for scalar functions $h \in L_2(\mathbf{R}^n)$, we have

$$e^{-tDB_0} E_0^+ \begin{bmatrix} h \\ 0 \end{bmatrix} = \nabla_A S_t h = \begin{bmatrix} \partial_{\nu_A} S_t h \\ \nabla_{\parallel} S_t h \end{bmatrix},$$

where S_t denotes the classical single layer potential operator

$$S_t h(x) := \int_{\mathbf{R}^n} \Gamma_{(0,y)}(t,x) h(y) dy, \qquad \text{with } \mathrm{div} A \nabla \Gamma_{(s,y)} = \delta_{(s,y)}.$$

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L_2 results for layer potentials

- Previously known: L_2-boundedness for small complex L_∞ perturbations of real equations.
  (Alfonseca-Auscher-Axelsson-Hofmann-Kim '08 + Hofmann-Kenig-Mayboroda-Pipher '12)
- Our result: L_2-boundedness for any divergence form system.
- When it does not exist as a singular integral operator, functional calculus defines the double layer potential operator, and gives the unique analytic continuation of the operator to general coefficients.
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Integration of the ODE with t-dependent coefficients

- $\partial_t f_t + DB_t f_t = 0, \quad$ for $f = \nabla_A u$
- $\partial_t f_t + DB_0 f_t = DB_0 \mathcal{E}_t f_t$, where $\mathcal{E}(t,x) = I - B(0,x)^{-1} B(t,x)$. Note $|\mathcal{E}(t,x)| \approx |A(t,x) - A(0,x)|$.
- $(\partial_t + \underbrace{DB_0}_{>0}) E_0^+ f_t = E_0^+ DB_0 \mathcal{E}_t f_t \quad$ and $\quad (\partial_t + \underbrace{DB_0}_{<0}) E_0^- f_t = E_0^- DB_0 \mathcal{E}_t f_t$
- $\lim_{t\to 0} f_t = f_0 \Rightarrow \quad E_0^+ f_t = e^{-tDB_0} E_0^+ f_0 + \int_0^t e^{-(t-s)DB_0} E_0^+ DB_0 \mathcal{E}_s f_s ds$
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- $f_t = e^{-tDB_0} E_0^+ f_0 + S_A \mathcal{E}_t f_t$

Definition

Define the maximal regularity operator S_A on functions $g : \mathbf{R}^{1+n}_+ \to \mathbf{C}^{1+n}$ by
$(S_A g)_t := \int_0^t DB_0 e^{-(t-s)DB_0} E_0^+ g_s ds - \int_t^\infty DB_0 e^{(s-t)DB_0} E_0^- g_s ds.$

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The operator $S_A\mathcal{E}$

- S_A is a "singular integral with operator valued kernel" since

$$\|DB_0 e^{(s-t)DB_0} E_0^{\pm}\|_{L_2\to L_2} \le C/|t-s|.$$

- $\mathcal{E}$ is essentially multiplication by $A(t,x) - A(0,x)$.
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- We are looking for a space $\mathcal{Z}$ of functions $\mathbf{R}^{1+n}_+ \to \mathbf{C}^{1+n}$ such that

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- For each of the considered boundary function spaces $L_2(\mathbf{R}^n)$ and $H^{-1}(\mathbf{R}^n)$ for $f|_{\mathbf{R}^n}$, we need an associated interior space $\mathcal{Z} \subset L_2^{\mathrm{loc}}(\mathbf{R}^{1+n}_+)$ for $f|_{\mathbf{R}^{1+n}_+}$.

The operator $S_A\mathcal{E}$

- S_A is a "singular integral with operator valued kernel" since

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Classical estimates on $\mathbf{R}^{1+n}_+$

Theorem (Carleson)

$$\iint_{\mathbf{R}^{1+n}_+} |h(t,x)||g(t,x)|dtdx \lesssim \int_{\mathbf{R}^n} Nh(x)Cg(x)dx$$

NT maximal functional: $(Nh)(x) := \sup_{|y-x|<s} |h(s,y)|$.
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NT maximal- and square function estimates for harmonic functions:

- Neumann:

$$\int_{\mathbf{R}^n} |\partial_\nu u|^2 dx \approx \int_{\mathbf{R}^n} |N(\nabla u)|^2 dx \approx \iint_{\mathbf{R}^{1+n}_+} |\nabla^2 u|^2 t dtdx.$$

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$$\int_{\mathbf{R}^n} |u|^2 dx \approx \int_{\mathbf{R}^n} |Nu|^2 dx \approx \iint_{\mathbf{R}^{1+n}_+} |\nabla u|^2 t dtdx.$$

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Interior function spaces for $f = \nabla_A u$

A natural space $\mathcal{Z}$ for f corresponding to $f_0 \in L_2(\mathbf{R}^n)$:

Definition

$\mathcal{X}^* := \{f : \mathbf{R}^{1+n}_+ \to \mathbf{C}^{1+n} \ ; \ N(W_2 f) \in L_2(\mathbf{R}^n)\}$,
where $W_2 f$ denotes the Whitney L_2 averaged function

$$(W_2 f)(t,x) := \left(\tfrac{1}{t^{1+n}} \iint_{W(t,x)} |f(s,y)|^2 ds dy \right)^{1/2},$$

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(Pre) dual spaces relative to $L_2(\mathbf{R}^{1+n}_+)$

Carleson's theorem yields:
$\iint_{\mathbf{R}^{1+n}_+} |g(t,x)||f(t,x)| dtdx \lesssim \int_{\mathbf{R}^n} C(g)N(f)dx \le \|C(g)\|_2 \|N(f)\|_2$.
With Whitney averages, we find the following duality.

Theorem (Hytönen, Rosén)

The NT space $\mathcal{X}^*$ is (non-reflexively) the dual space of $\mathcal{X} := \{g : \mathbf{R}^{1+n}_+ \to \mathbf{C}^{1+n} \; ; \; C(W_2 g) \in L_2(\mathbf{R}^n)\}$.

- At the level $f_0 \in L_2(\mathbf{R}^n)$, we have

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Boundedness of the singular integrals S_A

The maximal regularity operator

$$S_A g_t = \int_0^t DB_0 e^{-(t-s)DB_0} E_0^+ g_s ds - \int_t^\infty DB_0 e^{(s-t)DB_0} E_0^- g_s ds$$

is bounded on $L_2(\mathbf{R}^{1+n}_+; t^\alpha dtdx)$ for $|\alpha| < 1$ (corresponding to boundary Sobolev space $H^s(\mathbf{R}^n)$, $-1 < s < 0$), but not on $\mathcal{Y}$, $\mathcal{Y}^*$, $\mathcal{X}$ or $\mathcal{X}^*$.

Theorem (Auscher, Rosén)

The operator S_A has estimates

$$\|NW_2(S_A g)\|_2^2 \lesssim \iint_{\mathbf{R}^{1+n}_+} |g|^2 \frac{dtdx}{t}, \qquad \iint_{\mathbf{R}^{1+n}_+} |S_A g_t|^2 t dtdx \lesssim \|CW_2 g\|_2^2.$$

The operators $S_{A^*} : \mathcal{Y}^* \to \mathcal{X}^*$ and $S_A : \mathcal{X} \to \mathcal{Y}$ are adjoint relative to

$$\langle g, f \rangle := \iint_{\mathbf{R}^{1+n}_+} (-g_\perp + g_\parallel, B_0 f) dtdx.$$

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Sketch of proof

Consider the Hilbert space $\mathcal{H} := L_2(\mathbf{R}^{1+n}_+; t^\alpha dtdx)$ and DB_0 as a bisectorial operator in $\mathcal{H}$.
Define an $\mathcal{H}$ operator-valued holomorphic function $\lambda \mapsto F(\lambda)$ on $(-S_\omega) \cup S_\omega$:

$$F(\lambda) : g \mapsto \int_0^t \lambda e^{-(t-s)\lambda} \chi^+(\lambda) g_s ds - \int_t^\infty \lambda e^{(s-t)\lambda} \chi^-(\lambda) g_s ds.$$

- For $|\alpha| < 1$: $\sup_\lambda \|F(\lambda)\|_{\mathcal{H}\to\mathcal{H}} < \infty \Rightarrow S_A = F(DB_0)$ bounded.
- For $\alpha = -1$: write $F(\lambda) = F_0(\lambda) + F_1(\lambda)$ with

$$F_1(DB_0) = e^{tDB_0} \left(\int_0^\infty DB_0 e^{sDB_0} E_0^- g_s ds \right).$$

  Maximal- and square function estimates give bounds

$$F_1(DB_0) : \mathcal{Y}^* \to L_2(\mathbf{R}^n) \to \mathcal{X}^*.$$

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Multipliers vanishing on the boundary in a Carleson sense

If $S_A : \mathcal{X}^* \to \mathcal{X}^*$ had been bounded, it would have sufficed with $\|\mathcal{E}\|_\infty \approx \|A(t,x) - A(0,x)\|_\infty < \infty$. Now we need $\mathcal{E} : \mathcal{X}^* \to \mathcal{Y}^*$.

Carleson's theorem yields

$$\iint_{\mathbf{R}^{1+n}_+} |\mathcal{E}(t,x) f(t,x)|^2 \frac{dtdx}{t} \lesssim \int_{\mathbf{R}^n} C(\tfrac{\mathcal{E}^2}{t}) N(f^2) dx \le \|C(\tfrac{\mathcal{E}^2}{t})\|_\infty \|N(f)\|_2^2.$$

With Whitney averages, we find the following multiplier norm.

Theorem (Hytönen, Rosén)

The multiplier $f(t,x) \mapsto \mathcal{E}(t,x) f(t,x)$ has norms $\mathcal{X}^* \to \mathcal{Y}^*$ and $\mathcal{Y} \to \mathcal{X}$ equivalent to

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Main result 1: Cauchy formula for $\nabla_A u|_{\mathbf{R}^n} \in L_2(\mathbf{R}^n)$

Interior function space: $\mathcal{X}^* = \{f \ ; \ N(W_2 f) \in L_2(\mathbf{R}^n)\}$.
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$$\nabla_A u := (I - S_A \mathcal{E})^{-1} e^{-tDB_0} E_0^+ f_0$$

  constructs a function u with $\|N(W_2 \nabla_A u)\|_2 \lesssim \|f_0\|_2$, which is a weak solution to $\mathrm{div} A \nabla u = 0$, provided $\|A_t - A_0\|_*$ is sufficiently small.

- Any weak solution u with bound $\|N(W_2 \nabla_A u)\|_2 < \infty$ satisfies

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  From this one deduce limits
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