Dyadic harmonic analysis and weighted inequalities

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Weighted inequalities

Question (Two-weights L^p-inequalities for operator T)

Is there a constant $C_p(u,v) > 0$ such that

$$\|Tf\|_{L^p(v)} \leq C_p(u,v)\, \|f\|_{L^p(u)}, \ \textit{for all}\ \ f \in L^p(u)?$$

- *The weights u, v are a.e. positive locally integrable functions on $\mathbb{R}^d$.*
- $f \in L^p(u)$ *iff* $\|f\|_{L^p(u)} := (\int |f(x)|^p u(x)\, dx)^{1/p} < \infty$.
- *Operator* $T : L^p(u) \to L^p(v)$.

- Goal 1: given operator T, identify and classify weights u, v for whom the operator T is bounded from $L^p(u)$ to $L^p(v)$).
- Goal 2: understand nature of constant $C_p(u,v)$.

Latest results for Hilbert transform recall Ignacio's talk.

We concentrate on *one-weight L^2 inequalities*: $u = v = w$, and $p = 2$, for Calderón-Zygmund singular integral operators.

Question (One-weight L^2 inequality for operator T)

Is there a constant $C(w) > 0$ such that

$$\|Tf\|_{L^2(w)} \leq C(w)\,\|f\|_{L^2(w)}, \text{ for all } f \in L^2(w)?$$

We study one-weight inequalities in $L^2(w)$ for Calderón-Zygmund operators, and their commutators $[T, b] := Tb - bT$ with functions $b \in BMO$. More specifically, for simpler dyadic operators such as

- the martingale transform T_σ,
- Petermichl's Haar shift operator Ш ("Sha"),
- the dyadic paraproduct π_b,
- the dyadic square function S^d.

CZ operators are bounded in $L^p(w)$, when the weight w is in the Muckenhoupt A_p-class (Coifman-Fefferman '74), same holds for commutators (Alvarez-Bagby-Kurt-Pérez '96).

A_p weights

Definition

A weight w is in the *Muckenhoupt* A_p class if its A_p characteristic, $[w]_{A_p}$ is finite, where,

$$[w]_{A_p} := \sup_Q \left(\frac{1}{|Q|}\int_Q w\,dx\right)\left(\frac{1}{|Q|}\int_Q w^{-1/(p-1)}dx\right)^{p-1}, \quad 1 < p < \infty\,,$$

the supremum is over all cubes in $\mathbb{R}^d$ with sides parallel to the axes.

Note that a weight $w \in A_2$ if and only if

$$[w]_{A_2} := \sup_Q \left(\frac{1}{|Q|}\int_Q w\,dx\right)\left(\frac{1}{|Q|}\int_Q w^{-1}\,dx\right) < \infty.$$

Example

In $\mathbb{R}$, $w(x) := |x|^{\alpha}$, $w \in A_p \Leftrightarrow -1 < \alpha < p - 1$.

Commutators

Theorem (Chung, P., Pérez, Trans. AMS '12)

Given linear operator T, if for all $w \in A_2$ there exists a $C_n > 0$ such that and all $f \in L^2(w)$,

$$\|Tf\|_{L^2(w)} \leq C_n [w]_{A_2}^{\alpha} \|f\|_{L^2(w)}.$$

then its commutator with $b \in BMO$ obeys a quadratic bound,

$$\|[T,b]f\|_{L^2(w)} \leq C_n^* [w]_{A_2}^{\alpha+1} \|b\|_{BMO} \|f\|_{L^2(w)}.$$

- Proof uses classical Coifman-Rochberg-Weiss argument ('76) using the Cauchy integral formula.
- Generalizes to higher order comms. $T_b^k := [b, T_b^{k-1}]$ (power $\alpha + k$).
- These results are sharp for all $k \geq 1$ and all dimensions, as examples involving the Riesz transforms show, with $\alpha = 1$. Extrapolated bounds are sharp for all $1 < p < \infty$.

A_2 Conjecture

Transference theorem for commutators are useless unless there are operators known to obey the initial bound.
Do they exist? Yes, they do.

Theorem (Hytönen, Annals '12)

Let T be a Calderón-Zygmund operator, $w \in A_2$. Then there is a constant $C_{T,n} > 0$ such that for all $f \in L^2(w)$,

$$\|Tf\|_{L^2(w)} \leq C_{T,n} [w]_{A_2} \|f\|_{L^2(w)}.$$

As a corollary we conclude that for all such Ts,

$$\|[T,b]f\|_{L^2(w)} \leq C_{T,n} \|b\|_{BMO} [w]_{A_2}^2 \|f\|_{L^2(w)}.$$

$$\|[T_b^k f\|_{L^2(w)} \leq C_{T,n} \|b\|_{BMO} [w]_{A_2}^{1+k} \|f\|_{L^2(w)}.$$

Recent generalizations

- Extensions to commutators with fractional integral operators, two-weight problem Cruz-Uribe, Moen (Pub. Mat.'12).
- Extensions using $[w]_{A_1} \subset \cap_{p>1} A_p$ instead by Ortiz-Caraballo (Indiana '11).
- Hytönen, Pérez (arXiv '11) Prove mixed $A_2 - A_\infty$

$$\|[T,b]\|_{L^2(w)} \leq C_n \|b\|_{BMO} [w]_{A_2}^{\frac{1}{2}} \left([w]_{A_\infty} + [w^{-1}]_{A_\infty}\right)^{\frac{3}{2}}$$

  See also Ortiz-Caraballo, Pérez, Rela '12.
- On $L^r(w)$ with initial $[w]_{A_r}^{\alpha}$, and final $[w]_{A_r}^{\alpha+\max\{1,\frac{1}{r-1}\}}$ [P. '11].

Chronology of first Linear Estimates on $L^2(w)$

- *Maximal function* (Buckley Trans. AMS '93)
- *Martingale transform* (Wittwer 2000)
- *Dyadic square function* (Hukovic,Treil,Volberg OTAA'00; Wittwer MRL'00)
- *Beurling transform* (Petermichl, Volberg Duke '02)
- *Hilbert transform* (Petermichl 2003, AJM '07)
- *Riesz transforms* (Petermichl PAMS '08)
- *Dyadic paraproduct in* $\mathbb{R}$ (Beznosova JFA '08)

Estimates based on Bellman functions and bilinear Carleson estimates (except for maximal function).
The Bellman function method was introduced to harmonic analysis by Nazarov, Treil, Volberg. With their students and collaborators [Vasuynin, Slavin, Stokolos,...] have been able to use this method to obtain a number of astonishing results not only in this area [see Volberg's INRIA lecture notes '11 and references].

Sharp extrapolation

Theorem (Sharp Extrapolation Theorem)

If for all $w \in A_r$ there is $\alpha > 0$, and $C > 0$ such that

$$\|[Tf\|_{L^r(w)} \leq C[w]_{A_r}^{\alpha}\|f\|_{L^r(w)} \text{ for all } f \in L^r(w).$$

then for each $1 < p < \infty$ and for all $w \in A_p$, there is $C_{p,r} > 0$

$$\|[Tf\|_{L^p(w)} \leq C_{p,r}[w]_{A_p}^{\alpha \max\{1, \frac{r-1}{p-1}\}}\|f\|_{L^p(w)} \text{ for all } f \in L^p(w).$$

Dragičević, Grafakos, P., Petermichl '05, new proof Duoandikoetxea '11. Key are Buckley's sharp bounds ('93) for the maximal function

$$\|Mf\|_{L^p(w)} \leq C_p[w]_{A_p}^{\frac{1}{p-1}}\|f\|_{L^p(w)}.$$

Beautiful proof by Lerner '08, better $A_p - A_\infty$ estimates HytPz '11, extensions to homogeneous spaces HytKairema '10 (see Anna Kairema's talk).

Sharp extrapolation is not sharp

Example

Start with Buckley's sharp estimate on $L^r(w)$ for the maximal function, extrapolation will give sharp bounds only for $p < r$.

Example

Sharp extrapolation from $r = 2$, $\alpha = 1$, is sharp for the martingale, Hilbert, Beurling-Ahlfors and Riesz transforms for all $1 < p < \infty$ (for $p > 2$ Petermichl, Volberg '02, '07, '08; $1 \leq p < 2$ DGPPet).

Example

Extrapolation from linear bound in $L^2(w)$ is sharp for the dyadic square function only when $1 < p \leq 2$ ("sharp" DGPPet, "only" Lerner IJM'07). However, extrapolation from square root bound on $L^3(w)$ is sharp (Cruz-Uribe, Martell, Pérez, Adv. Math. '12)

Dyadic Harmonic Analysis: symmetries for H

- The Hilbert transform commutes with translations, dilations and anticommutes with reflections.
- A linear and bounded operator T on $L^2(\mathbb{R})$ that commutes with translations, dilations, and anticommutes with reflections must be a constant multiple of the Hilbert transform: $T = cH$.
- Using this principle, (Stefanie Petermichl 2000) showed that we can write H as a suitable "average of dyadic operators".

Dyadic intervals

Definition

The *standard dyadic intervals* $\mathcal{D}$ is the collection of intervals of the form $[k2^{-j}, (k+1)2^{-j})$, for all integers $k, j \in \mathbb{Z}$.

They are organized by generations: $\mathcal{D} = \cup_{j\in\mathbb{Z}}\mathcal{D}_j$, where $I \in \mathcal{D}_j$ iff $|I| = 2^{-j}$. Each generation is a partition of $\mathbb{R}$. They satisfy

Properties

- *Nested:* $I, J \in \mathcal{D}$ *then* $I \cap J = \emptyset, \quad I \subseteq J$, *or* $J \subset I$.
- *One parent: if* $I \in \mathcal{D}_j$ *then there is a unique interval* $\tilde{I} \in \mathcal{D}_{j-1}$ *(the parent) such that* $I \subset \tilde{I}$, *and* $|\tilde{I}| = 2|I|$.
- *Two children: There are exactly two disjoint intervals* $I_r, I_l \in \mathcal{D}_{j+1}$ *(the right and left children), such that* $I = I_r \cup I_l$, *and* $|I| = 2|I_r| = 2|I_l|$.

Random dyadic grids on $\mathbb{R}$

Definition

A dyadic grid in $\mathbb{R}$ is a collection of intervals, organized in generations, each of them being a partition of $\mathbb{R}$, that have the nestedness and two children per interval properties.

For example, the shifted and rescaled regular dyadic grid will be a dyadic grid. However these are NOT all possible dyadic grids. The following parametrization will capture ALL dyadic grids on $\mathbb{R}$.

Lemma

For each scaling or dilation parameter r with $1 \leq r < 2$, and the random parameter β with $\beta = \{\beta_i\}_{i\in\mathbb{Z}}$, $\beta_i = 0, 1$, let $x_j = \sum_{i<-j} \beta_i 2^i$, the collection of intervals $\mathcal{D}^{r,\beta} = \cup_{j\in\mathbb{Z}} \mathcal{D}_j^{r,\beta}$ is a dyadic grid. Where

$$\mathcal{D}_j^{r,\beta} := r\mathcal{D}_j^{\beta}, \qquad \textit{and} \qquad \mathcal{D}_j^{\beta} := x_j + \mathcal{D}_j.$$

Random dyadic grids were

- introduced by Nazarov, Treil and Volberg in their study of CZ singular integrals on non-homogeneous spaces [NTV 2003],
- utilized by Hytönen in his representation theorem [Hytonen 2012], see Hytönen, Kairema '10.

The advantage of this parametrization is that there is a very natural probability space, say $(\Omega, \mathbb{P})$ associated to the parameters, and averaging here means calculating the expectation in this probability space, that is $\mathbb{E}_\Omega f = \int_\Omega f(\omega)\, d\mathbb{P}(\omega)$.

Haar basis

Definition

Given an interval I, its associated *Haar function* is defined to be

$$h_I(x) := |I|^{-1/2}\big(\chi_{I_r}(x) - \chi_{I_l}(x)\big),$$

where $\chi_I(x) = 1$ if $x \in I$, zero otherwise.

- $\{h_I\}_{I\in\mathcal{D}}$ is a complete orthonormal system in $L^2(\mathbb{R})$ (Haar 1910).
- The Haar basis is an unconditional basis in $L^p(\mathbb{R})$ and in $L^p(w)$ if $w \in A_p$ (Treil-Volberg '96) for $1 < p < \infty$. Deduced from boundedness of the *martingale transform*

Definition (The Martingale transform)

$$T_\sigma f(x) := \sum_{I\in\mathcal{D}} \sigma_I \langle f, h_I\rangle h_I(x), \qquad \text{where} \quad \sigma_I = \pm 1.$$

The martingale transform is a good model for CZ singular operators.

Petermichl's dyadic shift operator

Definition

Petermichl's dyadic shift operator Ш (pronounced "Sha") associated to the standard dyadic grid $\mathcal{D}$ is defined for functions $f \in L^2(\mathbb{R})$ by

$$Шf(x) := \sum_{I \in \mathcal{D}} \langle f, h_I \rangle H_I(x),$$

where $H_I = 2^{-1/2}(h_{I_r} - h_{I_l})$.

- Ш is an isometry on $L^2(\mathbb{R})$, i.e. $\|Шf\|_2 = \|f\|_2$.
- Notice that $Шh_J(x) = H_J(x)$. The profiles of h_J and H_J can be viewed as a localized sine and cosine. First indication that the dyadic shift operator maybe a good dyadic model for the Hilbert transform.
- More evidence comes from the way the family $\{Ш_{r,\beta}\}_{(r,\beta)\in\Omega}$ interacts with translations, dilations and reflections.

Petermichl's representation theorem for H

Each dyadic shift operator does not have the symmetries that characterize the Hilbert transform, but an average over all random dyadic grids does.

Theorem (Petermichl C. R. Acad. Sci. Paris 2000)

$$\mathbb{E}_\Omega Ш_{r,\beta} = \int_\Omega Ш_{r,\beta} d\mathbb{P}(r,\beta) = cH,$$

- Result follows once one verifies that $c \neq 0$ (which she did!).
- $Ш_{r,\beta}$ are uniformly bounded on L^p $\Rightarrow$ Riesz's Theorem: H is bounded on L^p.
- Similar representation works for the *Beurling-Ahlfors* (Petermichl, Volberg Duke '02), *Riesz transforms* (Petermichl PAMS '08).
- There is a representation valid for ALL Calderón-Zygmund singular integral operators (Hytönen Annals '12).

Boundedness of H on weighted L^p

Theorem (Hunt, Muckenhoupt, Wheeden - Trans. AMS '73)

$$w \in A_p \Leftrightarrow \|Hf\|_{L^p(w)} \leq C_p(w)\|f\|_{L^p(w)}.$$

Dependence of the constant on $[w]_{A_p}$ was found 30 years later.

Theorem (Petermichl - Amer. J. Math. '07)

$$\|Hf\|_{L^p(w)} \leq C_p[w]_{A_p}^{\max\{1,\frac{1}{p-1}\}}\|f\|_{L^p(w)}.$$

Sketch of the proof.

For $p = 2$ suffices to find uniform (on the grids) linear estimates for Petermichl's shift operator on $L^2(w)$. For $p \neq 2$ sharp extrapolation automatically gives the result from the *linear estimate* on $L^2(w)$. □

Dyadic square function

Definition (The dyadic square function)

$$(S^d f)^2(x) := \sum_{I \in \mathcal{D}} \frac{|\langle f, h_I \rangle|^2}{|I|} \chi_I(x),$$

- S^d is an isometry on $L^2(\mathbb{R})$, bounded on $L^p(\mathbb{R})$ for $1 < p < \infty$,

$$\|S^d f\|_p \sim \|f\|_p \qquad (\text{"Plancherel" in } L^p).$$

It implies boundedness of T_σ (and Ш) on L^p

$$\|T_\sigma f\|_p \sim \|S^d(T_\sigma f)\|_p = \|S^d f\|_p \sim \|f\|_p.$$

$$(S^d(\text{Ш} f) = S^d f)$$

Dyadic square function - weighted estimates

- S^d is bounded on $L^2(w)$ if $w \in A_2$ (Buckley '93)

$$\|S^d f\|_{L^2(w)} \leq C[w]_{A_2}\|f\|_{L^2(w)} \quad \text{(HukTV, Wittwer '00)}.$$

$$\|f\|_{L^2(w)} \leq C[w]_{A_2}^{1/2}\|S^d f\|_{L^2(w)} \quad \text{(Petermichl, Pott '02)}.$$

  This reverse estimate leads to estimates $[w]_{A_2}^{3/2}$ for T_σ and Ш.
- Extensions to homogeneous spaces using Bellman functions. Doubling constant enters the estimate (P. '09).
- S^d is bounded on $L^p(w)$ if $w \in A_p$

$$\|S^d f\|_{L^p(w)} \leq C_p[w]_{A_p}^{\max\{\frac{1}{2},\frac{1}{p-1}\}}\|f\|_{L^p(w)} \quad \text{(CrMPz '10)}.$$

  This power is optimal. It corresponds to sharp extrapolation starting at $r = 3$ with square root power (see also Lerner, Wilson...).

Haar shift operators

Definition (Lacey, Reguera, Petermichl '10)

A *Haar shift operator of complexity* (m,n) is

$$Ш_{m,n}f(x) := \sum_{L\in\mathcal{D}} \sum_{I\in\mathcal{D}_m(L), J\in\mathcal{D}_n(L)} c^L_{I,J}\langle f, h_I\rangle h_J(x),$$

where the coefficients $|c^L_{I,J}| \leq \frac{\sqrt{|I|\,|J|}}{|L|}$, and $\mathcal{D}_m(L)$ denotes the dyadic subintervals of L with length $2^{-m}|L|$.

- The cancellation property of the Haar functions and the normalization of the coefficients ensures that $\|Ш_{m,n}f\|_2 \leq \|f\|_2$.
- T_σ is a Haar shift operator of complexity $(0,0)$.
- $Ш$ is a Haar shift operator of complexity $(0,1)$.
- The dyadic paraproduct π_b is not one of these.

The dyadic paraproduct

Definition

The *dyadic paraproduct* associated to $b \in BMO^d$ is

$$\pi_b f(x) := \sum_{I \in \mathcal{D}} m_I f \, \langle b, h_I \rangle h_I(x),$$

where $m_I f = \frac{1}{|I|} \int_I f(x)\, dx = \langle f, \chi_I / |I| \rangle$.

- Paraproduct and adjoint are bounded operators in $L^p(\mathbb{R})$ if and only if $b \in BMO^d$. (A locally integrable function $b \in BMO^d$ iff for all $J \in \mathcal{D}$ there is $C > 0$ such that $\int_J |b(x) - m_J b|^2 dx = \sum_{I \in \mathcal{D}(J)} |\langle b, h_I \rangle|^2 \leq C|J|$.)
- Formally, $fb = \pi_b f + \pi_b^* f + \pi_f b$.
- π_b bounded in $L^2(w)$ iff $w \in A_2$

$$\|\pi_b f\|_{L^2(w)} \leq C[w]_{A_2} \|f\|_{L^2(w)} \qquad \text{(Beznosova '08).}$$

Estimates for Shift Operators: before Hytönen's theorem

- Lacey, Petermichl, Reguera ('10) proved the A_2 conjecture for the Haar shift operators of arbitrary complexity (with constant depending exponentially in the complexity). Don't use Bellman functions. Use a *corona decomposition* and a *two-weight theorem for "well localized operators"* of NTV.
- Cruz-Uribe, Martell, Pérez ('10) recover all results for Haar shift operators. No Bellman functions, no two-weight results. Instead they use a local median oscillation introduced by Lerner. The method is very flexible, they get new results such as the sharp bounds for the square function for $p > 2$, for the dyadic paraproduct, also for vector-valued maximal operators, and two-weight results as well. Dependence on complexity is exponential.
- Pérez, Treil, Volberg ('10) get bound $[w]_{A_2} \log(1 + [w]_{A_2})$ for general T.

The A_2 conjecture (now Theorem)

Theorem (Hytönen 2010)

Let $1 < p < \infty$ *and let* T *be any Calderón-Zygmund singular integral operator in* $\mathbb{R}^n$*, then there is a constant* $c_{T,n,p} > 0$ *such that*

$$\|Tf\|_{L^p(w)} \le c_{T,n,p} \, [w]_{A_p}^{\max\{1,\frac{1}{p-1}\}} \|f\|_{L^p(w)}.$$

Sketch of the proof.

- Enough to show $p = 2$ thanks to sharp extrapolation.
- Prove a representation theorem in terms of Haar shift operators of arbitrary complexity and paraproducts on random dyadic grids.
- Prove linear estimates on $L^2(w)$ with respect to the A_2 characteristic for Haar shift operators and with polynomial dependence on the complexity (independent of the dyadic grid).

□

Hytönen's Representation theorem

Theorem (Hytönen's Representation Theorem 2010)

Let T be a Calderón-Zygmund singular integral operator, then

$$Tf = \mathbb{E}_\Omega \left(\sum_{(m,n)\in\mathbb{N}^2} a_{m,n} Ш^{r,\beta}_{m,n} f + \pi^{r,\beta}_{T1} f + (\pi^{r,\beta}_{T^*1})^* f \right),$$

with $a_{m,n} = e^{-(m+n)\alpha/2}$, α is the smoothness parameter of T.

- $Ш^{r,\beta}_{m,n}$ are Haar shift operators of complexity (m,n),
- $\pi^{r,\beta}_{T1}$ a dyadic paraproduct,
- $(\pi^{r,\beta}_{T^*1})^*$ the adjoint of the dyadic paraproduct ,

All defined on random dyadic grid $\mathcal{D}^{r,\beta}$.

Commutator $[H, b]$

Theorem (Daewon Chung '10)

$$\|[H,b]f\|_{L^2(w)} \leq C[w]_{A_2}^2 \|f\|_{L^2(w)}.$$

Daewon's "dyadic" proof is based on:

(1) the decomposition of the product bf

$$bf = \pi_b f + \pi_b^* f + \pi_f b$$

the first two terms are bounded in $L^p(w)$ when $b \in BMO$ and $w \in A_p$, the enemy is the third term.

(2) Use Petermichl's dyadic shift operator Ш instead of H,

$$[Ш, b]f = [Ш, \pi_b]f + [Ш, \pi_b^*]f + \left[Ш(\pi_f b) - \pi_{Ш f}(b)\right].$$

(3) Known linear bounds for paraproduct (Beznosova '08) and Ш (Petermichl '07).

cont. "dyadic proof" commutator

$$[Ш, b]f = [Ш, \pi_b]f + [Ш, \pi_b^*]f + [Ш(\pi_f b) - \pi_{Ш f}(b)].$$

- First two terms give quadratic bounds from the linear bounds for Ш and π_b, π_b^*.
- Boundedness of the commutator in $L^2(w)$ will be recovered from uniform boundedness of the third commutator.
- The third term is better, it obeys a linear bound, and so do halves of the other two commutators (Chung '09, using Bellman):

$$\|Ш(\pi_f b) - \pi_{Ш f}(b)\| + \|Ш\pi_b f\| + \|\pi_b^* Ш f\| \leq C\|b\|_{BMO}[w]_{A_2}\|f\|$$

- Providing uniform (sharp) quadratic bounds for commutator $[Ш, b]$ hence averaging

$$\|[H, b]\|_{L^2(w)} \leq C\|b\|_{BMO}[w]_{A_2}^2\|f\|_{L^2(w)}.$$

Known to be sharp, bad guys are the non-local terms $\pi_b Ш$, $Ш\pi_b^*$.

cont. "dyadic proof" commutator

- A posteriori one realizes the pieces that obey linear bounds are generalized Haar Shift operators and hence their linear bounds can be deduced from general results for those operators ...
- As a byproduct of Chung's dyadic proof we get that Beznosova's bounds for the paraproduct are optimal:

$$\|\pi_b f\|_{L^p(w)} \leq C_p [w]_{A_p}^{\max\{1, \frac{1}{p-1}\}} \|f\|_{L^p(w)}$$

Proof: by contradiction, if not for some p then $[H, b]$ will have better bound in $L^p(w)$ than the known optimal bound.

Recent progress

Active area of research! Hard to keep up with everything that is being posted in arXiv...

- There is a nice survey by Lacey up to '10.
- Since the appearance of Hytönen's theorem several simplifications of the argument have appeared [HytPzTV, NV, Treil, Lerner, Hyt, La, HytLaMartOrpReSawU-T, ... '11-12].
- There are already extensions to metric spaces with geometric doubling condition [NRezV '11, HytKa'11].
- Progress towards solution of Pérez's two weight bump conjecture [NRezTV, CrRezV, HytPz, Lerner ... '12].
- Also mixed $A_p - A_\infty$ estimates [HytPz, Lerner, HytLa '11, HytLaPz ...'12].
- Different attempts to get rid of one or more components of the proofs: randomness, Bellman functions, Haar shift operators [LaHyt, HytLaPz, Lerner...12]
- Role of positive dyadic shift operators [T,Hyt,La,Pz,La,Le'12].

Positive dyadic operators

- Cruz-Uribe, Martell, Pérez ‘10 showed in a few lines that

$$A_{S,\mathcal{D}}f(x) = \sum_{I\in\mathcal{D}} \chi_S(I)\, m_I f\, \chi_I(x)$$

  bounded when S is a "sparse" collection of dyadic intervals. These operators and maximal functions dominate Haar shifts.
- If $b \in BMO$ then $\pi_b^*\pi_b$ is a bounded positive operator.

$$\pi_b^*\pi_b f(x) = \sum_{I\in\mathcal{D}} \frac{b_I^2}{|I|} m_I f\, \chi_I(x),$$

- Generalized HSO are bounded iff Sawyer's testing conditions hold.
- Modern point of view is to control T or even better control maximal truncated version $T_\sharp$ by this positive operators, without using the representation theorem in terms of Haar shifts.

Thanks ;-)

;-)

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