A sharp bilinear estimate for the Klein-Gordon equation in two space-time dimensions*

Tohru Ozawa

Department of Applied Physics
Waseda University
Tokyo 169-8555, Japan

*** Joint Work with Keith Rogers, CSIC**

June 13, 2012, El Escorial, Madrid
9th International Conference on Harmonic Analysis and PDE

Blinear Estimates

- Wave Equation $(n \geq 2)$

$$||e^{it\sqrt{-\Delta}} f \cdot e^{it\sqrt{-\Delta}} g; L^2(\mathbb{R} \times \mathbb{R}^n)||^2$$
$$\leq C \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |\hat{f}(\xi)|^2 |\hat{g}(\xi)|^2 |\xi||\eta| (|\xi||\eta| - \xi \cdot \eta)^{\frac{n-3}{2}} d\xi d\eta$$

Klainerman and Machedon, CPAM ’93, Duke ’97
Bez and Rogers, J. Eur. Math. Soc., to appear

- Schrödinger Equation $(n \geq 1)$

$$||(-\Delta)^{-\frac{n-2}{2}} (e^{it\Delta} f \cdot \overline{e^{it\Delta} g}); L^2(\mathbb{R} \times \mathbb{R}^n)|| \leq C ||f; L^2(\mathbb{R}^n)|| \; ||g; L^2(\mathbb{R}^n)||$$

Tsutsumi and O, DIE ‘98

$$||e^{it\Delta} f \cdot e^{it\Delta} g; L^2(\mathbb{R} \times \mathbb{R}^n)||^2 \leq C \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |\hat{f}(\xi)|^2 |\hat{g}(\xi)|^2 |\xi - \eta|^{n-2} d\xi d\eta$$

$$(\mathrm{supp}\ \hat{f} \cap \mathrm{supp}\ \hat{g} = \emptyset \quad \text{if} \quad n = 1)$$

Carneiro, IMRN ‘09

Strichartz Estimates for the Klein-Gordon Equation

$$||(1-\Delta)^{-1/4}e^{it(1-\Delta)1/2}f;L^6(\mathbb{R}\times\mathbb{R})||\leq C||f;L^2(\mathbb{R})||$$

Segal, Adv. Math. ’76, Strichartz, Duke ‘77

Basic Idea

$$||e^{it(1-\Delta)1/2}f;L^6(\mathbb{R}\times\mathbb{R})||^2 = ||e^{it(1-\Delta)^{1/2}}f\cdot e^{it(1-\Delta)^{1/2}}f;L^3(\mathbb{R}\times\mathbb{R})||$$
$$\leq C||f;H^{1/2}(\mathbb{R})||^2$$

↑

Carleson and Sjölin, Studia Math. ’72
Fefferman, Acta. Math. ‘70

- Integral representation of $(e^{it(1-\Delta)1/2}f)^2$
- Suitable change of variables
- Estimate of the associated Jacobian
- Haussdorff – Young and Hardy – Littlewood – Sobolev inequalities

Goal : Bilinear estimate for the Klein-Gordon equation in 1+1 dimensions

A unified approach to the estimates of Segal, Carleson–Sjölin, Fefferman

Theorem 1. $\mathrm{supp}\ \hat{f} \cap \mathrm{supp}\ \hat{g} = \emptyset$

$$||e^{it(1-\Delta)^{1/2}} f \cdot e^{it(1-\Delta)^{1/2}} g; L^2(\mathbb{R} \times \mathbb{R})||^2 \leq \int_{\mathbb{R}} \int_{\mathbb{R}} |\hat{f}(\xi)|^2 |\hat{g}(\xi)|^2 \frac{(1+\xi^2)^{3/4}(1+\eta^2)^{3/4}}{|\xi - \eta|} \, d\xi d\eta$$

Remark. $\hat{f}(\xi) = (2\pi)^{-1/2} \int_{\mathbb{R}} \exp(-ix\xi) f(x) dx, \ \xi \in \mathbb{R}$

The inequality is sharp, but attained if and only if $f = 0$ or $g = 0$.

Theorem 2. $3p' \leq q \leq 4p', \ \frac{1}{2} < \frac{1}{p} + \frac{1}{q}$

$$||(1-\Delta)^{-1/p'} e^{it(1-\Delta)^{1/2}} f; L^q(\mathbb{R} \times \mathbb{R})|| \leq C||\hat{f}; L^p(\mathbb{R})||$$

Remark. $p = 2 \ \Rightarrow \ 6 \leq q \leq 8$

Proof of Theorem 1. $F(\xi,\eta)=\frac{1}{2}(\hat{f}(\xi)\hat{g}(\eta)+\hat{f}(\eta)\hat{g}(\xi))$

$$(e^{it(1-\Delta)^{1/2}}f)(x)\ \cdot\ (e^{it(1-\Delta)^{1/2}}g)(x)$$

$$=\frac{1}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}}F(\xi,\eta)\exp(ix(\xi+\eta)+it((1+\xi^2)^{1/2}+(1+\eta^2)^{1/2}))d\xi d\eta$$

$$=\frac{1}{\pi}\int\int_{\Omega}F(\xi,\eta)\exp(ix\underbrace{(\xi+\eta)}_{u}+it(\underbrace{(1+\xi^2)^{1/2}+(1+\eta^2)^{1/2}}_{v}))d\xi d\eta$$

$$\Omega=\{(\xi,\eta);\xi<\eta\}$$

$$T:\Omega\ni(\xi,\eta)\mapsto(u,v)\equiv(\xi+\eta,(1+\xi^2)^{1/2}+(1+\eta^2)^{1/2})\in T(\Omega)$$

$$=\frac{1}{\pi}\int\int_{T(\Omega)}F(T^{-1}(u,v))\exp(ixu+itv)|\det(T^{-1})'(u,v)|dudv$$

Fourier representation

$$\det(T^{-1})'(u,v)=1/\det T'(\xi,\eta)=1/(\eta(1+\eta^2)^{-1/2}-\xi(1+\xi^2)^{-1/2})$$

$$||e^{it(1-\Delta)^{1/2}} f \cdot e^{it(1-\Delta)^{1/2}} g \, ; L^2(\mathbb{R} \times \mathbb{R})||^2$$

$$= 4 \int \int_{T(\Omega)} |F(T^{-1}(u,v))|^2 |\det(T^{-1})'(u,v)|^2 du dv$$

Plancherel

$$= 4 \int \int_{\Omega} |F(\xi,\eta)|^2 \frac{1}{|\eta(1+\eta^2)^{-1/2} - \xi(1+\xi^2)^{-1/2}|} d\xi d\eta$$

$$= \int \int_{\Omega} (|\hat{f}(\xi)|^2 |\hat{g}(\eta)|^2 + |\hat{f}(\eta)|^2 |\hat{g}(\xi)|^2) \frac{1}{|\eta(1+\eta^2)^{-1/2} - \xi(1+\xi^2)^{-1/2}|} d\xi d\eta$$

Fourier support separation

Theorem 1 follows from :

$$(*)_1 : \quad 0 \leq \frac{\eta - \xi}{(1+\eta^2)^{3/4}(1+\xi^2)^{3/4}} \leq \frac{\eta}{(1+\eta^2)^{1/2}} - \frac{\xi}{(1+\xi^2)^{1/2}}, (\xi,\eta) \in \Omega$$

Proof of Theorem 2.

$$||e^{it(1-\Delta)^{1/2}}f; L^q(\mathbb{R}\times\mathbb{R})||^2 = ||e^{it(1-\Delta)^{1/2}}f \cdot e^{it(1-\Delta)^{1/2}}f; L^{q/2}(\mathbb{R}\times\mathbb{R})||$$

$$\underset{\text{HY}}{\leq} C\left(\int\int_{T(\Omega)}(|F(T^{-1}(u,v))||\det(T^{-1})'(u,v)|)^{(q/2)'}dudv\right)^{1/(q/2)'}$$

$$\leq C\left(\int\int_{\mathbb{R}^2}|\hat{f}(\xi)\hat{f}(\eta)^{(q/2)'}\frac{1}{|\eta(1+\eta^2)^{-1/2}-\xi(1+\xi^2)^{-1/2}|^{(q/2)'-1}}d\xi d\eta\right)^{1/(q/2)'}$$

$$\underset{(*)_\alpha}{\leq} C\left(\int\int_{\mathbb{R}^2}|\hat{f}(\xi)\hat{f}(\eta)|^{(q/2)'}\left(\frac{(1+\xi^2)^{(\alpha+2)/4}(1+\eta^2)^{(\alpha+2)/4}}{|\xi-\eta|^\alpha}\right)^{(q/2)'-1}d\xi d\eta\right)^{1/(q/2)'}$$

$$\underset{\text{H, HLS}}{\leq} C||(1+\xi^2)^{1/2p'}\hat{f}\ ;\ L^p(\mathbb{R})||^2, \quad \frac{1}{p'}=\frac{2+\alpha}{q}<\frac{2+\alpha}{2(1+\alpha)}$$

$$(*)_\alpha: \quad \frac{|\xi-\eta|^\alpha}{(1+\xi^2)^{(\alpha+2)/4}(1+\eta^2)^{(\alpha+2)/4}} \leq 2^{\alpha-1}\left|\frac{\eta}{(1+\eta^2)^{1/2}}-\frac{\xi}{(1+\xi^2)^{1/2}}\right|$$

$$(\xi,\eta)\in\mathbb{R}^2, \quad 1\leq\alpha\leq 2$$

Proof of $(*)_1$: $\cos\phi = \xi(1+\xi^2)^{-1/2},\ \cos\theta = \eta(1+\eta^2)^{-1/2},\ 0 \le \theta \le \phi \le \pi$

$$(\Rightarrow\ \ 1-\cos^2\phi = (1+\xi^2)^{-1},\ 1-\cos^2\theta = (1+\eta^2)^{-1})$$

$$\begin{aligned} 0 \le 2\sin\theta\sin\phi &= \cos(\phi-\theta) - \cos(\phi+\theta) && \text{Product-to-sum} \\ &\le 1 - \cos(\phi+\theta) \\ &= 2\sin^2\left(\frac{\phi+\theta}{2}\right) && \text{Power-reduction} \end{aligned}$$

$$\Rightarrow\ \ 0 \le (\sin\theta\sin\phi)^{1/2} \le \sin\left(\frac{\phi+\theta}{2}\right)$$

$$\begin{aligned} \Rightarrow\ \ 0 \le 2\sin\left(\frac{\phi-\theta}{2}\right)\cos\left(\frac{\phi-\theta}{2}\right)(\sin\theta\sin\phi)^{1/2} &\le 2\sin\left(\frac{\phi-\theta}{2}\right)(\sin\theta\sin\phi)^{1/2} \\ &\le 2\sin\left(\frac{\phi-\theta}{2}\right)\sin\left(\frac{\phi+\theta}{2}\right) \end{aligned}$$

$$\mathrm{RHS} = \cos\theta - \cos\phi \qquad \text{Product-to-sum}$$

$$\mathrm{LHS} = \sin(\phi-\theta)(\sin\theta\sin\phi)^{1/2} \qquad \text{Double-angle}$$

$$= (\sin\theta)^{1/2}(\sin\phi)^{3/2}\cos\theta - (\sin\theta)^{1/2}(\sin\phi)^{3/4}\cos\phi \qquad \text{Difference}$$

$$= (1-\cos^2\theta)^{1/2}(1-\cos^2\phi)^{3/4}\cos\theta - (1-\cos^2\theta)^{1/2}(1-\cos^2\phi)^{3/4}\cos\phi \qquad \text{Pythagorean}$$

Corollary to Theorem 2. $3q \le p' \le 4q, \quad \frac{1}{p} + \frac{1}{q} < \frac{3}{2}$

$$\left(\int_{\mathbb{R}} |\hat{F}(\xi, (1+\xi^2)^{1/2})|^q \frac{1}{(1+\xi^2)^{1/2}} d\xi \right)^{1/q} \le C||F; L^p(\mathbb{R}^2)||$$

Remark.

- Fourier restriction from $\mathbb{R}^2$ onto **hyperbola**

  $\downarrow$

  vanishing curvature at infinity

- Compact support version by Stein