Aspects of harmonic analysis related to hypersurfaces, and Newton diagrams
Part III

Detlef Müller
joint work with I. Ikromov

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B. Fourier restriction: Adapted coordinates

Assume that

$$S = \{(x_1, x_2, \phi(x_1, x_2)) : (x_1, x_2) \in \Omega\},$$

where $(0,0) \in \Omega$ and $\phi(0,0) = 0$, $\nabla\phi(0,0) = 0$.

Theorem (Ikromov, M.)

Assume that there is a linear coordinate system adapted to ϕ, where ϕ is smooth of finite type. If the support of $\rho \geq 0$ is contained in a sufficiently small neighborhood of 0, then

$$\Big(\int_S |\widehat{f}|^2 \rho d\sigma\Big)^{1/2} \leq C_p \|f\|_{L^p(\mathbb{R}^3)}, \qquad f \in \mathcal{S}(\mathbb{R}^3), \tag{1.1}$$

for $1 \leq p \leq p_c$, where $p_c' := 2h(\phi) + 2$.

Remarks:

1. Knapp type examples show that our result is sharp.
2. **A. Magyar** had obtained partial results in the analytic case before.

On the proof

Let $h = h(\phi), \nu = \nu(\phi)$, and recall from Part A that $d\mu = \rho d\sigma$ satisfies the estimate

$$|\widehat{d\mu}(\xi)| \leq C \, \|\rho\|_{C^3(S)} \, (\log(2 + |\xi|))^{\nu} (1 + |\xi|)^{-1/h} \tag{1.2}$$

1. If $\nu = 0$, the theorem follows directly from (1.2) and

Theorem (Greenleaf)

Assume that $\widehat{\mu}(\xi) \lesssim |\xi|^{-1/h}$. Then the restriction estimate

$$\Big(\int_S |\widehat{f}|^2 \, d\mu \Big)^{1/2} \leq C_p \|f\|_{L^p}$$

holds for every $p \geq 1$ such that $p' \geq 2h + 2$.

2. The endpoint $p' = 2h(\phi) + 2$ can be obtained by Littlewood-Paley theory.

A useful variant of the Stein-Tomas argument

Let me sketch a variant of the classical proof of Greenleaf's theorem (for $\mathbb{R}^n$), which closely ties with dyadic frequency decompositions and whose method turned out to be particularly useful when the coordinates are not adapted to ϕ.

For $\lambda \gg 1$, define μ^λ by

$$\widehat{\mu^\lambda}(\xi) = \chi_1\Big(\frac{\xi}{\lambda}\Big)\hat{\mu}(\xi),$$

where χ is again supported in an annulus $\mathcal{A}$ so that (ignoring small frequencies) $\mu = \sum_{j\geq 0} \mu^{2^j}$. Writing $x = (x', x_n)$, then

$$\begin{aligned} \mu^\lambda(x) &= \lambda^n \int \check{\chi}_1(\lambda(x'-y), \lambda(x_n - \phi(y')))\eta(y')\,dy' \\ &= \lambda \int \check{\chi}_1\Big(z, \lambda(x_n - \phi(x' - \frac{z}{\lambda}))\Big)\eta(x' - \frac{z}{\lambda})\,dz \end{aligned}$$

From $\widehat{\mu}(\xi) \lesssim |\xi|^{-1/h}$ and this formula we derive that

$$\|\widehat{\mu^\lambda}\|_\infty \lesssim \lambda^{-1/h}, \qquad \|\mu^\lambda\|_\infty \lesssim \lambda.$$

Let

$$Tf := f * \hat{\mu}, \qquad T^\lambda f := f * \widehat{\mu^\lambda},$$

hence

$$T = \sum_{j \geq 0} T^{2^j}$$

Since

$$\int_S |\hat{f}|^2 \, d\mu \leq \|T\|_{p \to p'} \|f\|_p^2,$$

we need to estimate $\|T\|_{p \to p'}$.
Interpolating the estimates above we get

$$\|T^\lambda\|_{p \to p'} \lesssim \lambda^{-\frac{1-\theta(h+1)}{h}}, \qquad \text{if } \theta = \frac{2}{p'} \tag{1.3}$$

Note: If $p' = p'_c := 2h + 2$, where $\theta = 1/(h+1)$, then

$$\|T^\lambda\|_{p_c \to p'_c} \lesssim 1.$$

Define analytic family of operators $T_z f := f * \widehat{\mu_z}, \quad 0 \le \mathrm{Re}\, z \le 1$, where

$$\mu_z := -(1 - 2^{(1-z)(1+1/h)}) \sum_{j=0}^{\infty} 2^{\frac{1-z(h+1)}{h} j} \mu^{2^j}$$

Note that $T_{1/(h+1)} = T$. By (1.3) and disjointness of Fourier supports, we get

$$\|\widehat{\mu_{it}}\|_\infty \lesssim 1, \qquad t \in \mathbb{R}.$$

By Stein's interpolation theorem, we are left to prove that

$$\|\mu_{1+it}\|_\infty \lesssim 1, \qquad t \in \mathbb{R}. \tag{1.4}$$

Put $\nu^\lambda := \lambda^{-1}\mu^\lambda$, and $\nu_j := \nu^{2^j}$. Then

$$\nu^\lambda(x) = \int F(\lambda, x, z)\, dz, \tag{1.5}$$

where

$$F(\lambda, x, z) := \check{\chi}_1\Big(z, \lambda(x_n - \phi(x' - \frac{z}{\lambda}))\Big)\eta(x' - \frac{z}{\lambda}),$$

and

$$\mu_{1+it} = -(1 - 2^{-it\frac{h+1}{h}})\sum_{j=0}^{\infty} 2^{-it\frac{h+1}{h}j}\nu_j$$

And: Suppose $\check{\chi}_1$ had compact support. Then $|z| \lesssim 1$, and

$$\lambda(x_n - \phi(x' - \frac{z}{\lambda})) = \lambda(x_n - \phi(x')) + r(\lambda, x', z),$$

where

$$\begin{aligned} |r(\lambda, x', z)| &\lesssim |z|, \\ |\lambda\partial_\lambda r(\lambda, x', z)| &\lesssim \frac{|z|^2}{\lambda}. \end{aligned} \tag{1.6}$$

This shows that necessarily $|\lambda(x_n - \phi(x')| \lesssim 1$. By summation by parts in

$$\mu_{1+it} = -\frac{1}{2}(1 - 2^{-it\frac{h+1}{h}}) \sum_{j=0}^{\infty} 2^{-it\frac{h+1}{h}j} \nu_j,$$

we may ess. estimate

$$|\mu_{1+it}(x)| \lesssim \sum_{j=0}^{\infty} |\nu_j(x) - \nu_{j+1}(x)|,$$

where

$$\nu_j(x) - \nu_{j+1}(x) = \int_1^2 \int (\lambda\partial_\lambda F)(s\lambda, x, z)\, dz\, ds.$$

The passage to the differences $\nu_j(x) - \nu_{j+1}(x)$ thus ess. allows to replace F in the definition of ν^λ by $\lambda\partial_\lambda F$! This produces extra factors (compared to F) of the form

$$\lambda(x_n - \phi(x')), \quad \lambda\partial_\lambda r(\lambda, x', z) = O(|z|^2/\lambda), \ldots,$$

which then allow to sum over all dyadic $\lambda = 2^j$, with a bound independent of x, since $|\lambda(x_n - \phi(x'))| \lesssim 1$.

The case where $\nu(\phi) = 1$

To capture the endpoint $p = p_c$ when $\nu = 1$, we cannot apply Greenleaf's result directly. But, in the last lecture, we had effectively decomposed

$$\mu = \sum_{k=k_0}^{\infty} \mu_k, \quad \text{where} \quad \mu_k = (\chi_k \otimes 1)\mu,$$

and shown that

$$|\widehat{\mu_k}(\xi)| = |J_k(\xi)| \leq C 2^{-k|\kappa|}(1 + 2^{-k}|\xi_3|)^{-1/h}$$

(no logarithmic factor yet!) Greenleaf's result can then be used to show (by re-scaling) that

$$\int |\hat{f}(x)|^2 \, d\mu_k(x) \leq C^2 \|f\|_{p_c}^2 \qquad \forall k \geq k_0 \tag{1.7}$$

Choose again $\tilde{\chi} \in C_0^\infty(\mathbb{R}^2)$ supported in an annulus $\mathcal{A}$ so that $\tilde{\chi} = 1$ on the support of χ, and define dyadic frequency decomposition operators Δ'_k by

$$\widehat{\Delta'_k f}(x) := \tilde{\chi}(\delta_{2^k} x')\, \hat{f}(x', x_3)$$

Then $\int |\hat{f}(x)|^2 d\mu_k(x) = \int |\widehat{\Delta'_k f}(x)|^2 d\mu_k(x)$, so that (1.7) implies

$$\int |\hat{f}(x)|^2 d\mu_k(x) \le C^2\, \|\widehat{\Delta'_k f}\|_{p_c}^2,$$

for any $k \ge k_0$. In combination with Minkowski's inequality, this implies

$$\begin{aligned}\int |\hat{f}(x)|^2 d\mu(x) &= \sum_{k\ge k_0} \int |\hat{f}(x)|^2 d\mu_k(x) \le C^2 \sum_{k\ge k_0} \|\Delta'_k f\|_{p_c}^2 \\ &\le C^2 \left\| \left(\sum_{k\ge k_0} |\Delta'_k f(x)|^2 \right)^{1/2} \right\|_{p_c}^2,\end{aligned}$$

since $p_c < 2$. We conclude by means of Littlewood-Paley theory.

B. Fourier restriction: Non-adapted coordinates

Assume next that there is no linear coordinate system which is adapted to ϕ.

We may then assume that there are adapted coordinates y of the form $y_1 = x_1, y_2 = x_2 - \psi(x_1)$, where

$$\psi(x_1) = x_1^m \omega(x_1), \quad \text{with } \omega(0) \neq 0 \text{ and } m \geq 2. \tag{2.1}$$

ϕ^a will again denote ϕ when expressed in these adapted coordinates. We use the notions introduced for the study of the Newton polyhedron $\mathcal{N}(\phi^a)$ of ϕ^a from the previous lecture.

r-height

Let

$$\Delta^{(m)} := \{(t, t+m+1) : t \in \mathbb{R}\}.$$

For any edge $\gamma_l \subset L_l := \{(t_1, t_2) \in \mathbb{R}^2 : \kappa_1^l t_1 + \kappa_2^l t_2 = 1\}$ of $\mathcal{N}(\phi^a)$ define h_l by

$$\Delta^{(m)} \cap L_l = \{(h_l - m, h_l + 1)\},$$

i.e.,

$$h_l = \frac{1 + m\kappa_1^l - \kappa_2^l}{\kappa_1^l + \kappa_2^l}, \tag{2.2}$$

Define the restriction height, or short, r-height, of ϕ by

$$h^r(\phi) := \max(d, \max_{\{l=1,\dots,n+1 : a_l > m\}} h_l).$$

Remarks:

1. For L in place of L_l and κ in place of κ^l, one has $m = \kappa_2/\kappa_1$ and $d = 1/(\kappa_1 + \kappa_2)$, so that one gets d in place of h_l in (2.2).
2. Since $m < a_l$, we have $h_l < 1/(\kappa_1^l + \kappa_2^l)$, hence $h^r(\phi) \le h(\phi)$.

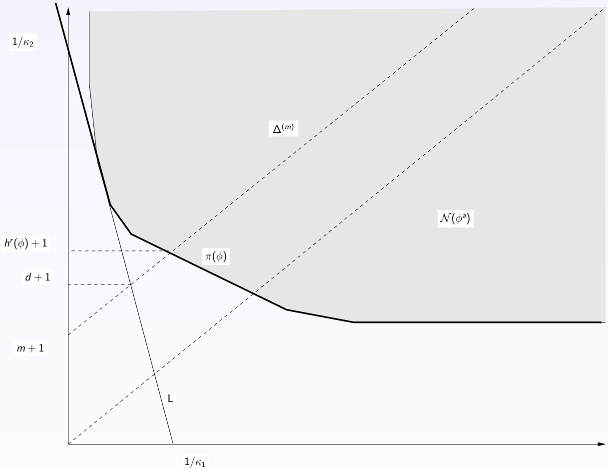

Figure: r-height

Theorem (Ikromov, M.)

Let $\phi \neq 0$ be real analytic, and assume that there is no linear coordinate system adapted to ϕ. If the support of $\rho \geq 0$ is contained in a sufficiently small neighborhood of 0, then the Fourier restriction estimate (1.1), i.e.,

$$\Big(\int_S |\widehat{f}|^2\, d\mu\Big)^{1/2} \leq C_p \|f\|_{L^p},$$

holds true for every $p \geq 1$ such that $p' \geq p'_c := 2h^r(\phi) + 2$.

Remarks:

- An application of Greenleaf's result would imply, at best, that the condition $p' \geq 2h(\phi) + 2$ is sufficient for (1.1) to hold, which is a strictly stronger condition than $p' \geq p'_c$.
- It can be shown that the number m is well-defined, i.e., it does not depend on the chosen linearly adapted coordinate system x.

Example 2

$$\phi(x_1, x_2) := (x_2 - x_1^m)^n, \qquad n, m \geq 2.$$

The coordinates (x_1, x_2) are not adapted. Adapted coordinates are $y_1 := x_1, y_2 := x_2 - x_1^m$, in which ϕ is given by

$$\phi^a(y_1, y_2) = y_2^n.$$

Here

$$\kappa_1 = \frac{1}{mn}, \quad \kappa_2 = \frac{1}{n},$$

$$d := d(\phi) = \frac{1}{\kappa_1 + \kappa_2} = \frac{nm}{m+1} < n,$$

and

$$p'_c = \begin{cases} 2d+2, & \text{if } n \leq m+1, \\ 2n, & \text{if } n > m+1 \ . \end{cases}$$

On the other hand, $h := h(\phi) = n$, so that $2h + 2 = 2n + 2 > p'_c$.

Remarks:

- An analogous theorem holds true even for smooth, finite type functions ϕ, under an additional Condition (R) which, roughly speaking, requires that whenever the Newton diagram suggests that a root with leading term given by the principal root jet $\psi(x_1)$ should have multiplicity B, then indeed such a root of multiplicity B does exist (this is a condition on the behavior of flat terms). Condition (R) is always satisfied when ϕ is real-analytic.
- **Examples:** Condition (R) holds true for

$$\phi_g(x_1, x_2) = (x_2 - x_1^2 - f(x_1))^2,$$

for every flat smooth function $f(x_1)$ (i.e., $f^{(j)}(0) = 0$ for every $j \in \mathbb{N}$), but fails for

$$\phi_b(x_1, x_2) := (x_2 - x_1^2)^2 + f(x_1),$$

unless f vanishes identically.

- There is a more invariant description of the notion of r-height, somewhat in the spirit of Varchenko's definition of height

Necessity of the condition $p' \geq 2h^r(\phi) + 2$

Let γ_l be any edge of $\mathcal{N}(\phi^a)$ with $a_l > m$, and choose the weight κ^l such that γ_l lies on the line L_l given by $\kappa_1^l t_1 + \kappa_2^l t_2 = 1$. Consider the region

$$D_\varepsilon^a := \{y \in \mathbb{R}^2 : |y_1| \leq \varepsilon^{\kappa_1^l}, |y_2| \leq \varepsilon^{\kappa_2^l}\}, \quad \varepsilon > 0,$$

in adapted coordinates y. In the original coordinates x, it corresponds to

$$D_\varepsilon := \{x \in \mathbb{R}^2 : |x_1| \leq \varepsilon^{\kappa_1^l}, |x_2 - \psi(x_1)| \leq \varepsilon^{\kappa_2^l}\}.$$

Assume that ε is sufficiently small. Since

$$\phi^a(\varepsilon^{\kappa_1^l} y_1, \varepsilon^{\kappa_2^l} y_2) = \varepsilon \left(\phi_{\kappa^l}^a(y_1, y_2) + O(\varepsilon^\delta)\right)$$

for some $\delta > 0$, we have that $|\phi^a(y)| \leq C\varepsilon$ for every $y \in D_\varepsilon^a$, i.e.,

$$|\phi(x)| \leq C\varepsilon \quad \text{for every } x \in D_\varepsilon. \tag{2.3}$$

Moreover, for $x \in D_\varepsilon$,

$$|x_2| \leq \varepsilon^{\kappa_2^l} + |\psi(x_1)| \lesssim \varepsilon^{\kappa_2^l} + \varepsilon^{m\kappa_1^l}.$$

Since $m \leq a_l = \kappa_2^l/\kappa_1^l$, we find that

$$|x_2| \lesssim \varepsilon^{m\kappa_1^l},$$

so that we may assume that D_ε is contained in the box where $|x_1| \leq \varepsilon^{\kappa_1^l}, |x_2| \leq \varepsilon^{m\kappa_1^l}$. Choose f_ε such that

$$\widehat{f_\varepsilon}(x_1, x_2, x_3) = \chi_0\Big(\frac{x_1}{\varepsilon^{\kappa_1^l}}\Big)\chi_0\Big(\frac{x_2}{\varepsilon^{m\kappa_1^l}}\Big)\chi_0\Big(\frac{x_3}{\varepsilon}\Big).$$

Then by (2.3) we see that $\widehat{f_\varepsilon}(x_1, x_2, \phi(x_1, x_2)) \geq 1$ on D_ε, hence, if $\rho(0) \neq 0$, then

$$\Big(\int_S |\widehat{f_\varepsilon}|^2 \,\rho d\sigma\Big)^{1/2} \geq |D_\varepsilon|^{1/2} = \varepsilon^{(\kappa_1^l+\kappa_2^l)/2}.$$

Since $\|f_\varepsilon\|_p \simeq \varepsilon^{((1+m)\kappa_1^l+1)/p'}$, we find that the restriction estimate can hold true only if

$$p' \geq 2\frac{(1+m)\kappa_1^l + 1}{\kappa_1^l + \kappa_2^l} = 2h_l + 2,$$

where we recall that $h_l = \frac{1+m\kappa_1^l-\kappa_2^l}{\kappa_1^l+\kappa_2^l}$.

Notice that the argument still works if we replace the previous line L_l by the line L associated to the weight κ, and $\phi^a_{\kappa^l}$ by ϕ^a_κ. Since here $m\kappa_1 = \kappa_2$, this leads to the condition $p' \geq 2d + 2$, so that altogether necessarily

$$p' \geq 2\max(d, \max_{l : a^l > m} h_l) + 2 = 2h^r(\phi) + 2.$$

Q.E.D.

Sufficiency of the condition $p' \geq 2h^r(\phi) + 2$: I. Key steps in the proof when $d > 5/2$

- In problem A, it had been natural to distinguish between the cases where $h < 2$ and where $h \geq 2$, since in the latter case, in many situations a reduction to a one-dimensional situation had been possible by means of the van der Corput lemma.
- Problem B turns out to be of different nature, and we shall distinguish between the cases where $d > 5/2$ and where $d < 5/2$. The latter case turns out to be the most difficult one.

So, assume first that $d > 5/2$. Write $h^r := h^r(\phi)$.

Step 1: Reduction to a narrow neighborhood of the principal root

As in Problem A, localize to the narrow κ-homogeneous subdomain

$$|x_2 - b_1 x_1^m| \leq \varepsilon x_1^m, \tag{2.4}$$

Indeed, the technique of proof that we used in the case of adapted coordinates can essentially be carried over to the domain complementary to (2.12) without major new ideas, since one can show that the Fourier transforms of the corresponding dyadic pieces μ_k of the measure μ satisfy estimates of the form

$$|\widehat{\mu_k}(\xi)| \leq C 2^{-k|\kappa|}(1 + 2^{-k}|\xi_3|)^{-1/d}.$$

Recall that $h^r \geq d$.

Step 2: Domain decomposition into "homogeneous" domains D_l and transition domains E_l.

Assume again that the principal face of the Newton polyhedron of ϕ^a is a **compact edge.**

Following the scheme from the previous lecture, we narrow down the domain (2.12) to the neighborhood $D_{\mathrm{pr}} := D_\lambda$ of the principal root jet given by (2.5), where

$$|x_2 - \psi(x_1)| \leq N_\lambda x_1^{a_\lambda} \tag{2.5}$$

by decomposing the difference set of the domains (2.12) and (2.5) (up to some remainder E_{l_0-1}) into the domains ($l = l_0, \dots, \lambda - 1$)

$$\begin{aligned} D_l &:= \{(x_1, x_2) : \varepsilon_l x_1^{a_l} < |x_2 - \psi(x_1)| \leq N_l x_1^{a_l}\}, \\ E_l &:= \{(x_1, x_2) : N_{l+1} x_1^{a_{l+1}} < |x_2 - \psi(x_1)| \leq \varepsilon_l x_1^{a_l}\} \end{aligned}$$

Contribution by the domains E_l.

Denote by μ_{E_l} the contribution of the transition domains E_l to the measure μ. Decompose μ_{E_l} bi-dyadically w.r. to the adapted coordinates y as

$$\mu_{E_l} = \sum_{j,k} \mu_{j,k},$$

so that $\mu_{j,k}$ is supported where $y_1 = x_1 \sim 2^{-j}$ and $y_2 = x_2 - \psi(x_1) \sim 2^{-k}$. Observe that this a curved rectangle in the original coordinates x.

Goal: Try again to use Littlewood-Paley theory in order to reduce to uniform restriction estimates for the family of measure $\mu_{j,k}$, i.e.,

$$\int_S |\widehat{f}|^2 \, d\mu_{j,k} \le C \|f\|^2_{L^p}, \qquad \forall j, k, \tag{2.6}$$

for $p \le p_c$.

- **Problem:** Because of the non-linearity $\psi(x_1)$, this is not possible by Littlewood-Paley techniques in the variables x_1 and x_2!
- **Good news:** We can use the variables x_1 and x_3!.

- Indeed

$$\phi^a(y) = c_l\, y_1^{A_l} y_2^{B_l}\Big(1 + \text{small error}\Big) \quad \text{on} \quad E_l^a,$$

- $\Longrightarrow$ On E_l^a respectively E_l (E_l^a represents E_l in the adapted coordinates y!) the conditions $y_1 \sim 2^{-j}, y_2 \sim 2^{-k}$ are equivalent to the conditions

$$x_1 \sim 2^{-j} \quad \text{and} \quad \phi(x) \sim 2^{-(A_l j + B_l k)}$$

- Re-scale the measures $\mu_{j,k}$ to get normalized measures $\nu_{j,k}$ supported on a surface $S_{j,k}$ where $y_1 \sim 1 \sim y_2$. One finds that $S_{j,k}$ is a small perturbation of the limiting surface

$$S_\infty := \{(y_1,\, y_1^m \omega(0),\, c y_1^{A_l} y_2^{B_l}) : y_1 \sim 1 \sim y_2\},$$

But $|\partial(c y_1^{A_l} y_2^{B_l})/\partial y_2| \sim 1$, since $B_l \geq 1$, which shows that S_∞, and hence also $S_{j,k}$, is a smooth hypersurface with one non-vanishing principal curvature (with respect to y_1) of size ~ 1.

$$\implies \quad |\widehat{\nu_{j,k}}(\xi)| \le C(1+|\xi|)^{-1/2},$$

uniformly in j and k. Applying Greenleaf's restriction theorem to these measures, and scaling these estimates back, we eventually arrive (in a not completely trivial way) at (2.6). It is important to observe here that Greenleaf's result implies restriction estimates for

$$p' \ge 2(1+2) = 6,$$

which is sufficient for our purposes, since $p_c' \ge 2d + 2 > 2(5/2) + 2 > 6$.

Contribution by the domains D_l.

- Dyadic decomposition of D_l in adapted coordinates y by means of the κ^l-dilations + re-scaling leads to re-scaled measure ν_k corresponding to the measures μ_k :

$$\langle \nu_k,\, f\rangle := \int f(y_1,\, 2^{(m\kappa_1^l-\kappa_2^l)k} y_2 + y_1^m \omega(2^{-\kappa_1^l k} y_1),\, \phi^k(y))\, \tilde\eta(y)\, dy$$

- Finite partition of unity allows to assume that $\tilde\eta$ is supported in a thin set $U(c_0)$, on which

$$y_1 \sim 1 \quad \text{and} \quad |y_2 - c_0 y_1^{a_l}| \le \varepsilon y_1^{a_l}.$$

Then ν_k is supported in a variety S_k which in the limit as $k \to \infty$ tends to the variety

$$S_\infty := \{g_\infty(y_1, y_2) := (y_1,\, \omega(0) y_1^m\,, \phi^a_{\kappa^l}(y)) : (y_1, y_2) \in U(c_0)\},$$

since $m\kappa_1^l - \kappa_2^l < a_l \kappa_1^l - \kappa_2^l = 0$ and since ϕ^k tends to $\phi^a_{\kappa^l}$. Here, c_0 is fixed with $|c_0| \le N_l$.

We need uniform restriction estimates for the family of measures ν_k!
Depending on c_0, different cases may arise. Recall

$$S_\infty := \{g_\infty(y_1, y_2) := (y_1,\, \omega(0)y_1^m\,, \phi^a_{\kappa^l}(y)) : (y_1, y_2) \in U(c_0)\},$$

1. Case. $\partial_2 \phi^a_{\kappa^l}(1, c_0) \neq 0.$ Use $z_2 := \phi^a_{\kappa^l}(y_1,\, y_2)$ in place of y_2 as a new coordinate for S_∞ (which thus is a hypersurface). $\Longrightarrow$ Since $y_1 \sim 1$ on $U(c_0)$, we find that S_∞, hence also S_k, is a hypersurface with one non-vanishing principal curvature. Argue then as for the domains E_l.

2. Case. $\partial_2 \phi^a_{\kappa^l}(1, c_0) = 0$, but $\partial_1 \phi^a_{\kappa^l}(1, c_0) \neq 0.$

- Since $\phi^a_{\kappa^l}$ is a κ^l-homogenous polynomial, Euler's homogeneity relation implies that $\phi^a_{\kappa^l}(1,\, c_0) \neq 0$.
- Fibre the variety S_∞ into the family of curves

$$\gamma_c(y_1) := g_\infty(y_1, cy_1^{a_l}) = (y_1, \omega(0)y_1^m, \phi^a_{\kappa^l}(y_1, cy_1^{a_l})),$$

  for c sufficiently close to c_0.
- $\gamma_{c_0}(y_1) = (y_1, \omega(0)y_1^m, b_0 y_1^{1/\kappa_1^l})$, where $b_0 \neq 0$, has non-vanishing torsion. The same applies then to the curves γ_c, and for k sufficiently large, we do obtain the analogous results for the varieties S_k.

This allows to decompose the measure $d\nu_k$ as a direct integral of measures $d\Gamma_c$ supported on curves γ_c^l with non-vanishing torsion. We may thus apply Drury's Fourier restriction theorem for curves with non-vanishing torsion to the measures $d\Gamma_c$:

$$\Big(\int |\hat{f}|^q \, d\Gamma_c\Big)^{\frac{1}{q}} \le C_{p,q}\|f\|_{L^p(\mathbb{R}^3)}, \quad p' > 7, q \ge p'/6.$$

Since we assume

$$p'_c \ge 2(d+1) > 2(5/2+1) = 7,$$

these estimates, after re-scaling to the measures μ_k, yield the desired restriction estimates for the contributions by the domains D_l.

Notice: it is here that we need the condition $d = h_{\rm lin} > 5/2$

3. Case. $\partial_2 \phi^a_{\kappa^l}(1, c_0) = 0$ and $\partial_1 \phi^a_{\kappa^l}(1, c_0) = 0$. Then $\phi^a_{\kappa^l}(1, c_0) = 0$ (Euler), hence $\phi^a_{\kappa^l}$ has a real root of multiplicity $B \geq 2$ at $(1, c_0)$, thus

$$\phi^a_{\kappa^l}(y_1, y_2) = y_2^{B_l}(y_2 - c_0 y_1^{a_l})^B Q(y_1, y_2), \tag{2.7}$$

where Q is a κ^l-homogenous, $Q(1, c_0) \neq 0$ and $Q(1, 0) \neq 0$. One can also prove that $B < d/2$.

Follow the Stein-Tomas method outlined earlier. Localize to frequencies of size $\Lambda > 1$:

$$\widehat{\nu_k^\Lambda}(\xi) := \chi_1\Big(\frac{\xi}{\Lambda}\Big)\widehat{\nu_k}(\xi).$$

Claim:

$$\|\widehat{\nu_k^\Lambda}\|_\infty \leq C\Lambda^{-1/B}; \tag{2.8}$$

$$\|\nu_k^\Lambda\|_\infty \leq C\Lambda^{2-1/B}. \tag{2.9}$$

Indeed, the first estimate follows easily by applications of van der Corput's lemma.

As for the second, in the limit as $k \to \infty$, ν_k^Λ is given by

$$\begin{aligned} &\nu_\infty^\Lambda(x_1, x_2, x_3) \\ = \ & \Lambda^3 \int (\mathcal{F}^{-1}\chi_1)(\Lambda(x_1 - y_1), \Lambda(x_2 - \omega(0)y_1^m), \Lambda(x_3 - \phi^a_{\kappa^l}(y_1, y_2))\,\tilde{\eta}(y)\,dy_1 \\ = \ & \Lambda^2 \int (\mathcal{F}^{-1}\chi_1)(z_1, \Lambda(x_2 - \omega(0)(x_1 - \frac{z_1}{\Lambda})^m), \Lambda(x_3 - \phi^a_{\kappa^l}(x_1 - \frac{z_1}{\Lambda}, y_2)) \\ & \qquad \eta_1(x_1 - \frac{z_1}{\Lambda}, y_2)\,dz_1\,dy_2, \end{aligned}$$

where η_1 localizes again to $U(c_0)$. Since $|\partial_2^B \phi^a_{\kappa^l}(y_1, y_2))| \simeq 1$ on the domain of integration, sublevel estimates of van der Corput type imply that the integral with respect to y_2 can be estimated by $O(\Lambda^{-1/B})$.

Interpolating the estimates (2.8) and (2.9), and applying the Stein-Tomas argument, one finds that one can even sum the corresponding estimates over all dyadic $\Lambda \gg 1$ and obtains

$$\Big(\int |\widehat{f}|^2\, d\nu_k\Big)^{1/2} \le C_p \|f\|_{L^p} \qquad \text{if } p' > 4B.$$

But, $p_c' \ge 2d+2 > 4B$, since $B < d/2$. Scaling back to the measures μ_k, we find

$$\Big(\int |\widehat{f}|^2\, d\mu_k\Big)^{1/2} \le C_p \|f\|_{L^p}, \qquad k \ge k_0,$$

provided $p' \ge 2h_l + 2$. This applies to p_c, since $h^r(\phi) \ge h_l$.

Observe: the dyadic decomposition into the measures μ_k can be achieved by dyadic decomposition in the variable x_1, so that these uniform estimates allow to sum over all k by means of Littlewood-Paley theory applied to variable x_1!

Step 3: Contribution by the domain $D_{\mathrm{pr}} = D_\lambda$ containing the principal root jet

What remains to be understood is the contribution by the domain $D_{\mathrm{pr}} = D_\lambda$ given by

$$|x_2 - \psi(x_1)| \leq N_\lambda x_1^{a_\lambda}.$$

- Here, the condition $B < d/2$ will in general no longer be true, not even the weaker condition $B < h^r/2$, as examples shows!
- Only in Case 3 where $\nabla \phi^a_{\mathrm{pr}}(1, c_0) = 0$, we used $B < d/2$; in all other cases we can essentially argue as before.

Stopping time argument to produce further domain decomposition:

- Put $\phi^{(1)} := \phi^a$. If Case 3 does not appear for any choice of c_0, then we stop our algorithm with $\phi^{(1)}$, and are done.
- If Case 3 applies to c_0, so that $c_0 y_1^{a\lambda}$ is a root of $\phi^a_{\kappa^\lambda}$, say of multiplicity $M_1 \geq 2$, then we define new coordinates z in place of y by putting

$$z_1 := x_1 \qquad \text{and} \quad z_2 := x_2 - \psi(x_1) - c_0 x_1^{a\lambda}, \tag{2.10}$$

  and express ϕ by $\phi^{(2)}$ in the coordinates z. Again, if Case 3 does not appear (for $\phi^{(2)}$ in place of $\phi^{(1)}$) in the corresponding z-domain, we stop our algorithm.
- Otherwise, we iterate this step.

This algorithm eventually leads to a further domain decomposition of D_{pr} into "homogeneous" domains $D_{(l)}$ and transition domains $E_{(l)}$, which can eventually be treated by methods similar to those applied for the domains E_l and D_l.

Sufficiency of the condition $p' \geq 2h^r(\phi) + 2$: II. Some ideas of the proof when $d \leq 5/2$

Fact: Here $h^r(\phi) = d$, so that $p'_c = 2d + 2$. Assume even $d = h_{\mathrm{lin}} < 2$.

Theorem (Normalforms (Arnol'd, Duistermaat, Sirsma, Ikromov/M.))

If $d < 2$, then locally ϕ is of the form

$$\phi(x_1, x_2) = b(x_1, x_2)(x_2 - \psi(x_1))^2 + b_0(x_1). \tag{2.11}$$

Here b, b_0 and ψ are smooth, and ψ is again the principal root jet, and either

(a) *$b(0,0) \neq 0$, and either b_0 is flat (singularity of type A_∞), or of finite type n, i.e., $b_0(x_1) = x_1^n \beta(x_1)$, where $\beta(0) \neq 0$ (singularity of type A_{n-1});*

or

(b) *$b(0,0) = 0$ and $b(x_1, x_2) = x_1 b_1(x_1, x_2) + x_2^2 b_2(x_2)$, with $b_1(0,0) \neq 0$ (singularity of type D).*

Assume type A_{n-1}.

1. Step: Employ the normal form in order to estimate certain two-dimensional oscillatory integrals that arise in estimating the Fourier transforms of surface carried measures, reduce again to the domain

$$|x_2 - b_1 x_1^m| \leq \varepsilon x_1^m, \tag{2.12}$$

2. Step: Dyadic decomposition + re-scaling by means of the κ-dilations (associated to $\pi(\phi)$) we may reduce to a phase function

$$\phi(x,\delta) := b(\delta_1 x_1, \delta_2 x_2)\Big(x_2 - x_1^m \omega(\delta_1 x_1)\Big)^2 + \delta_0 x_1^n \beta(\delta_1 x_1), \tag{2.13}$$

where $\delta = (\delta_0,\, \delta_1,\, \delta_2) = (2^{-(n\kappa_1 - 1)k}, 2^{-\kappa_1 k}, 2^{-\kappa_2 k})$ are small parameters, and $b(\delta_1 x_1, \delta_2 x_2) \sim b(0,0) \neq 0,\ \beta(0) \neq 0$.

What we then need to prove is the following

Proposition

Given any point $v = (v_1, v_2)$ such that $v_1 \sim 1$ and $v_2 = v_1^m \omega(0)$, there exists a neighborhood V of v in $(\mathbb{R}_+)^2$ such that for every cut-off function $\eta \in \mathcal{D}(V)$, the measure ν_δ given by

$$\langle \nu_\delta, f \rangle := \int f(x, \phi(x, \delta))\, \eta(x_1, x_2)\, dx$$

satisfies a restriction estimate

$$\Big(\int |\widehat{f}|^2 \, d\nu_\delta \Big)^{1/2} \leq C_{p,\eta} \|f\|_{L^p(\mathbb{R}^3)},$$

whenever $p' \geq 2d + 2$, provided δ is sufficiently small.

Littlewood-Paley theory in x_3 allows to reduce to uniform restriction estimates for the following family of measures

$$\langle \nu_{\delta,j}, f\rangle := \int f(x,\phi(x,\delta))\,\chi(2^{2j}\phi(x,\delta))\eta(x_1,x_2)\,dx,$$

namely

$$\Big(\int |\widehat{f}|^2\,d\nu_{\delta,j}\Big)^{1/2} \le C_{p,\eta}\|f\|_{L^p(\mathbb{R}^3)}. \tag{2.14}$$

If $2^{2j}\delta_0 \ll 1$, then this localization means in fact again a localization to a curved rectangle where $|x_1 - v_1| < \varepsilon$ and $|x_2 - x_1^m\omega(\delta_1 x_1)| \sim 2^{-j}$, but in other cases, it has another meaning.

Refined spectral decomposition: for every triple $\Lambda = (\lambda_1,\lambda_2,\lambda_3)$ of dyadic numbers $\lambda_i = 2^{-k_i} \ge 1$, define ν_j^Λ by

$$\widehat{\nu_j^\Lambda}(\xi) = \chi_1\Big(\frac{\xi_1}{\lambda_1}\Big)\chi_1\Big(\frac{\xi_2}{\lambda_2}\Big)\chi_1\Big(\frac{\xi_3}{\lambda_3}\Big)\widehat{\nu_{\delta,j}}(\xi), \tag{2.15}$$

so that $\nu_{\delta,j} = \sum_\Lambda \nu_j^\Lambda$, where summation is essentially over all these dyadic triples Λ.

- For Λ, follow again the Stein-Tomas approach, by estimating $\|\widehat{\nu_j^\Lambda}\|_\infty$ and $\|\nu_j^\Lambda\|_\infty$.
- Distinguish various cases, depending on the relative sizes of λ_1, λ_2 and λ_3.
- Most difficult case: where $\lambda_1 \sim \lambda_2 \sim \lambda_3$, and $2^{2j}\delta_0 \sim 1$.

Theorem

Let ϕ be of type A_{n-1}, with $m = 2$ and finite $n \geq 5$. Then

$$\sum_{2\leq\lambda_1\sim\lambda_2\sim\lambda_3\leq 2^{6j}} \int_S |\widehat{f}|^2 \, d\nu_j^\Lambda \leq C\, 2^{\frac{1}{7}j} \, \|f\|^2_{L^{14/11}(\mathbb{R}^3)}, \tag{2.16}$$

for all $j \in \mathbb{N}$ sufficiently big, say $j \geq j_0$, where the constant C does neither depend on δ, nor on j.

Double-Airy type analysis

Proof requires yet further refinements.

Indeed, the Fourier transform of ν_j^Λ is an oscillatory integral with complete phase

$$\begin{aligned}\Phi(y;\delta,j,\xi) &= \xi_1 y_1 + \xi_2 y_1^2\omega(\delta_1 y_1) + \xi_3\sigma y_1^n\beta(\delta_1 y_1)\\ &\quad +2^{-j}\xi_2 y_2 + \xi_3 b^\sharp(y,\delta,j)\, y_2^2.\end{aligned}$$

Here

$$\sigma := 2^{2j}\delta_0 \sim 1, \qquad |b^\sharp(x,\delta,j)| \sim 1.$$

- If $|\xi_1| \sim |\xi_2| \sim |\xi_2|$, then ϕ may have degenerate critical points, with non-vanishing third derivatives, with respect to the variable x_1, as well as x_2, so that we encounter oscillatory integrals of "double Airy type".
- This case requires a further dyadic frequency decomposition with respect to the distance to certain "Airy cones," in combination with subtle variants of the complex interpolation method described earlier, in order to capture also the endpoint $p = p_c = 14/11$.

THANKS

FOR YOUR

ATTENTION!