Aspects of harmonic analysis related to hypersurfaces, and Newton diagrams
Part II

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A. Decay of the Fourier transform of the surface measure μ: Outline of some main ideas

Recall that $\widehat{\mu}(\xi)$ as an oscillatory integral

$$\widehat{\mu}(\xi) =: J(\xi) = \int_{\Omega} e^{-i(\xi_3\phi(x_1,x_2)+\xi_1 x_1+\xi_2 x_2)}\eta(x)\, dx, \quad \xi \in \mathbb{R}^3,$$

$\eta \in C_0^\infty(\Omega)$, where ϕ is smooth, finite type, and $\phi(0,0) = 0, \nabla\phi(0,0) = 0$.

Theorem (Ikromov, M.)

Let $S = \mathrm{graph}(\phi)$, ϕ smooth and finite type. Then there exists a neighborhood $U \subset S$ of $x^0 = 0$ such that for every $\rho \in C_0^\infty(U)$ the following estimate holds true for every $\xi \in \mathbb{R}^3$:

$$|\widehat{d\mu}(\xi)| \leq C\, \|\rho\|_{C^3(S)}\, (\log(2+|\xi|))^{\nu(\phi)}(1+|\xi|)^{-1/h(\phi)} \tag{1.1}$$

Remarks:

1. The case $h(\phi) < 2$ is covered by Duistermaat's work.
2. It can also be handled by means of certain normal forms for ϕ which will be discussed later.

We shall therefore subsequently assume that $h := h(\phi) \geq 2$.

Often, estimates can be reduced to one-dimensional ones and an application of van der Corput's lemma, respectively

Lemma (Björk; Arhipov)

Let $f \in C^\infty(I, \mathbb{R})$ be of polynomial type $n \geq 2$ $(n \in \mathbb{N})$, i.e.,

$$0 < c_1 \leq \sum_{j=2}^{n} |f^{(j)}(s)| \leq c_2 \quad \textit{for every } s \in I.$$

Then

$$\Big| \int_I e^{i\lambda f(s)} g(s)\, ds \Big| \leq C \|g\|_{C^1(I)} (1 + |\lambda|)^{-1/n},$$

where the constant C depends only on the constants c_1 and c_2.

The case where the coordinates are adapted to ϕ

Here

$$d = d(\phi) = h = h(\phi).$$

Assume that the principal face $\pi(\phi)$ is a compact edge.
We may assume that the integration in $J(\xi)$ takes place over the half-space $\mathbb{R}^2_+$ where $x_1 > 0$.
Recall: If κ is the principal weight, then $\phi_{\mathrm{pr}} = \phi_\kappa$ is δ_r-homogeneous of degree 1, where $\delta_r(x_1, x_2) = (r^{\kappa_1} x_1, r^{\kappa_2} x_2)$.
Choose $\chi \in C_0^\infty(\mathbb{R}^2)$ supported in an annulus $\mathcal{A}$ on which $|x| \sim 1$, such that the functions $\chi_k := \chi \circ \delta_{2^k}$ form a dyadic partition of unity, and decompose

$$J(\xi) = \sum_{k=k_0}^{\infty} J_k(\xi),$$

where k_0 is sufficiently large, with

$$J_k(\xi) := \int_{\mathbb{R}^2_+} e^{-i(\xi_3 \phi(x) + \xi_1 x_1 + \xi_2 x_2)} \eta(x) \chi_k(x)\, dx.$$

Scaling by $\delta_{2^{-k}}$ yields

$$J_k(\xi) = 2^{-k|\kappa|} \int_{\mathbb{R}^2_+} e^{-i\left(2^{-k}\xi_3\phi^k(x)+2^{-k\kappa_1}\xi_1x_1+2^{-k\kappa_2}\xi_2x_2\right)} \eta(\delta_{2^{-k}}(x))\chi(x)\, dx, \tag{1.2}$$

with $\phi^k(x) := 2^k\phi(\delta_{2^{-k}}x)$. Notice that

$$\phi^k(x) = \phi_\kappa(x) + \text{error term}.$$

Claim: For every $x^0 \in \mathcal{A}$, there exist a unit vector $e \in \mathbb{R}^2$ and $j \in \mathbb{N}$ with $2 \le j \le h$ such that $\partial_e^j \phi_\kappa(x^0) \ne 0$.

Proof:

- if $\nabla\phi_\kappa(x^0) \ne 0$, then the homogeneity of ϕ_κ and Euler's homogeneity relation imply that $\operatorname{rank}(D^2\phi_\kappa(x^0)) \ge 1$, so we may choose $j = 2$, for a suitable vector e.
- if $\nabla\phi_\kappa(x^0) = 0$, then by Euler's homogeneity relation $\phi_\kappa(x^0) = 0$ as well. Thus the function ϕ_κ vanishes in x^0 of order $j \ge 2$. This implies that $j \le m(\phi_{\mathrm{pr}}) \le d = h$, in view of our characterization of adaptedness. The claim follows.

Apply van der Corput's lemma to the integration along lines parallel to the direction e in the integral defining $J_k(\xi)$ near the point x^0. Fubini's theorem and a partition of unity argument then yields

$$\begin{aligned} |J_k(\xi)| &\leq C\, 2^{-k|\kappa|}(1+2^{-k}|\xi_3|)^{-1/j} \\ &\leq C\, 2^{-k|\kappa|}(1+2^{-k}|\xi|)^{-1/M}, \end{aligned} \tag{1.3}$$

where M denotes the maximal j arising in this context.
Summation in k :

$$|J(\xi)| \leq C \begin{cases} (1+|\xi|)^{-1/M}, & \text{if } M|\kappa| > 1 \ , \\ (1+|\xi|)^{-1/M} \log(2+|\xi|), & \text{if } M|\kappa| = 1 \ , \\ (1+|\xi|)^{-|\kappa|}, & \text{if } M|\kappa| < 1 \ . \end{cases} \tag{1.4}$$

Since $\pi(\phi)$ is a compact edge, $1/|\kappa| = d = h$, and moreover $M \leq d$. This implies $|\kappa| M \leq 1$. Recall also that $\nu(\phi) = 1$ if and only if $M = m(\phi_{\rm pr}) = h$, i.e., if and only if $M|\kappa| = 1$, we obtain estimate (1.1).

The case where the coordinates are not adapted to ϕ

Step 1: Reduction to a narrow neighborhood of the principal root.
Away from the principal root of $\phi_{\rm pr}$, we can argue in the same way as before, since the multiplicity of any real root of $\phi_{\rm pr}$ different from the principal root is bounded by $d \le h$. I.e., we can reduce to a narrow κ-homogeneous neighborhood of the curve $x_2 = b_1 x_1^m$, of the form

$$|x_2 - b_1 x_1^m| \le \varepsilon x_1^m, \tag{1.5}$$

by means of a function $\rho_1(x) := \chi_0((x_2 - b_1 x_1^m)/(\varepsilon x_1^m))$, where χ_0 is a suitable smooth bump function supported in the interval $[-1, 1]$ and $\varepsilon > 0$ is sufficiently small. I.e., in place of $J(\xi)$, it suffices to estimate $J^{\rho_1}(\xi)$, where we write

$$J^\chi(\xi) := \int_{\mathbb{R}_+^2} e^{-i(\xi_3\phi(x_1,x_2)+\xi_1x_1+\xi_2x_2)}\eta(x)\,\chi(x)\,dx$$

if χ is any integrable function.

Step 2: Domain decompositions into "homogeneous" domains D_l and transition domains E_l.

Change to the adapted coordinates y :

$$J^{\rho_1}(\xi) = \int_{\mathbb{R}^2_+} e^{-i(\xi_3 \phi^a(y_1,y_2) + \xi_1 y_1 + \xi_2 \psi(y_1) + \xi_2 y_2)} \tilde{\eta}(y)\, \tilde{\chi}_0\Big(\frac{y_2}{\varepsilon y_1^m}\Big)\, dy. \qquad (1.6)$$

Edges and weights associated to $\mathcal{N}(\phi^a)$:

- vertices (A_l, B_l), $l = 0, \dots, n$, where $A_{l-1} < A_l$, $l = 1, \dots, n$,
- edges $\gamma_l := [(A_{l-1}, B_{l-1}), (A_l, B_l)], l = 1, \dots, n$. The unbounded horizontal edge with left endpoint (A_n, B_n) will be denoted by γ_{n+1}.
- weight associated to γ_l : $\kappa^l = (\kappa^l_1, \kappa^l_2)$ is so that

$$\gamma_l \subset L_l := \{(t_1, t_2) \in \mathbb{R}^2 : \kappa^l_1 t_1 + \kappa^l_2 t_2 = 1\}.$$

- exponents $a_l := \frac{\kappa^l_2}{\kappa^l_1}, \quad l = 1, \dots, n; \ a_{n+1} := \infty.$

If $l \leq n$, the κ^l-principal part $\phi^a_{\kappa^l}$ of ϕ^a corresponding to the supporting line L_l is of the form

$$\phi^a_{\kappa^l}(y) = c_l\, y_1^{A_{l-1}} y_2^{B_l} \prod_\alpha \Big(y_2 - c_l^\alpha y_1^{a_l}\Big)^{N_\alpha} \qquad (1.7)$$

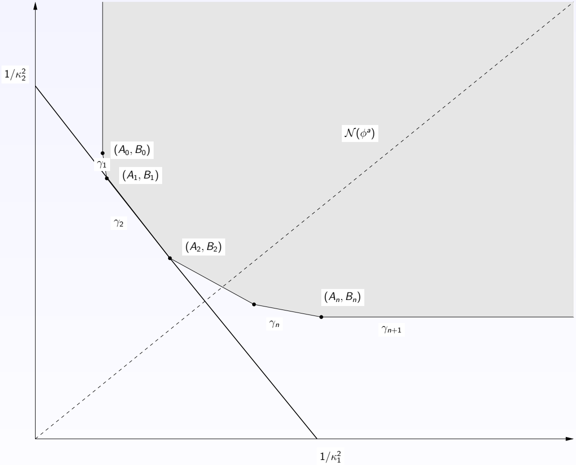

Figure: 3. Edges and weights

Relation with Puiseux series expansions of roots

Assume ϕ is analytic. Then

$$\phi^a(y_1, y_2) = U(y_1, y_2) y_1^{\nu_1} y_2^{\nu_2} \prod_r (y_2 - r(y_1)),$$

where the r denote the non-trivial roots $r = r(y_1)$ of ϕ^a and $U(0,0) \neq 0$. These roots locally admit Puiseux series expansions

$$r(y_1) = c_{l_1}^{\alpha_1} y_1^{a_{l_1}} + c_{l_1 l_2}^{\alpha_1 \alpha_2} y_1^{a_{l_1 l_2}^{\alpha_1}} + \cdots + c_{l_1 \cdots l_p}^{\alpha_1 \cdots \alpha_p} y_1^{a_{l_1 \cdots l_p}^{\alpha_1 \cdots \alpha_{p-1}}} + \cdots ,$$

where

$$c_{l_1 \cdots l_p}^{\alpha_1 \cdots \alpha_{p-1} \beta} \neq c_{l_1 \cdots l_p}^{\alpha_1 \cdots \alpha_{p-1} \gamma} \quad \text{for} \quad \beta \neq \gamma,$$

$$a_{l_1 \cdots l_p}^{\alpha_1 \cdots \alpha_{p-1}} > a_{l_1 \cdots l_{p-1}}^{\alpha_1 \cdots \alpha_{p-2}},$$

with strictly positive exponents $a_{l_1 \cdots l_p}^{\alpha_1 \cdots \alpha_{p-1}} > 0$ and non-zero complex coefficients $c_{l_1 \cdots l_p}^{\alpha_1 \cdots \alpha_p} \neq 0$. . The leading exponents in these series are the numbers

$$a_1 < a_2 < \cdots < a_n.$$

Clusters of roots

Group the roots into the clusters $[l], l = 1, \dots, n$, where the l'th cluster $[l]$ consistes of all roots with leading exponent a_l.

Note: If $\delta_s^l(x_1, x_2) = (s^{\kappa_1^l} x_1, s^{\kappa_2^l} x_1),\ s > 0$, denote the κ^l-dilations, and if $r \in [l_1]$, then for $y = (y_1, y_2)$ in a bounded set

$$\delta_s^l y_2 = s^{\kappa_2^l} y_2, \quad r(\delta_s^l y_1) = s^{a_{l_1} \kappa_1^l} c_{l_1}^{\alpha_1} y_1^{a_{l_1}} (1 + O(s^\varepsilon))$$

as $s \longrightarrow 0$, for some $\varepsilon > 0$. Consequently, since $\kappa_2^l / \kappa_1^l = a_l$,

$$\delta_s^l y_2 - r(\delta_s^l y_1) = (1 + O(s^\varepsilon)) \begin{cases} -s^{a_{l_1} \kappa_1^l} c_{l_1}^{\alpha_1} y_1^{a_{l_1}}, & \text{if } l_1 < l\ , \\ s^{\kappa_2^l} (y_2 - c_l^{\alpha_l} y_1^{a_l}), & \text{if } l_1 = l\ , \\ s^{\kappa_2^l} y_2, & \text{if } l_1 > l, \end{cases}$$

$$\phi_{\kappa^l}^a = C_l y_1^{\nu_1 + \sum_{l_1 < l} |[l_1]| a_{l_1}} y_2^{\nu_2 + \sum_{l_1 > l} |[l_1]|} \prod_{\alpha_1} (y_2 - c_l^{\alpha_1} y_1^{a_l})^{N_{l,\alpha_1}}, \tag{1.8}$$

where N_{l,α_1} denotes the number of roots in the cluster $[l]$ with leading term $c_l^{\alpha_1} y_1^{a_l}$.

Note that $\prod_{\alpha_1}(y_2 - c_l^{\alpha_1} y_1^{a_l})^{N_{l,\alpha_1}} = (\phi_{[l]})_{\kappa^l}$. Moreover,

$$\nu_1 + \sum_{l_1<l} |[l_1]| a_{l_1} = A_{l-1}, \ \nu_2 + \sum_{l_1>l} |[l_1]| = B_l.$$

Comparing this with (1.7), the close relation between the Newton polyhedron of ϕ^a and the Pusieux series expansion of roots becomes evident, and accordingly we say that the edge $\gamma_l := [(A_{l-1}, B_{l-1}), (A_l, B_l)]$ is associated to the cluster of roots $[l]$.

Choose integer $l_0 \geq 1$ such that

$$a_1 < \cdots < a_{l_0-1} \leq m < a_{l_0} < \cdots < a_l < a_{l+1} < \cdots < a_n.$$

Since the original coordinates x were assumed to be non-adapted, the vertex (A_{l_0-1}, B_{l_0-1}) will lie strictly above the bisectrix, i.e., $A_{l_0-1} < B_{l_0-1}$, Assume that the principal face $\pi(\phi^a)$ is a compact edge. Assume also that that

$$m(\phi^a_{\rm pr}) < d(\phi^a), \qquad \text{hence } \nu(\phi) = 0,$$

since otherwise, we may run Varchenko's algorithm one more step so that $\pi(\phi^a)$ becomes a vertex.

Choose $\lambda > l_0$ so that the edge $\gamma_\lambda = [(A_{\lambda-1}, B_{\lambda-1}), (A_\lambda, B_\lambda)]$ is the principal face $\pi(\phi^a)$ (cf. Figure 3, where $\lambda = 3$.)
We shall narrow down the domain (1.5), $|x_2 - b_1 x_1^m| \le \varepsilon x_1^m$, to a neighborhood D_λ of the principal root jet of the form

$$|x_2 - \psi(x_1)| \le N_\lambda x_1^{a_\lambda}, \tag{1.9}$$

where N_λ is a constant to be chosen later. This domain is κ^λ-homogeneous in the adapted coordinates y.

Subdomains

Decompose the difference set of the domains (1.5) and (1.9) (up to some remainder E_{l_0-1}) into the domains ($l = l_0, \dots, \lambda - 1$)

$$\begin{aligned} D_l &:= \{(x_1, x_2) : \varepsilon_l x_1^{a_l} < |x_2 - \psi(x_1)| \le N_l x_1^{a_l}\}, \\ E_l &:= \{(x_1, x_2) : N_{l+1} x_1^{a_{l+1}} < |x_2 - \psi(x_1)| \le \varepsilon_l x_1^{a_l}\} \end{aligned}$$

The $\varepsilon_l > 0$ are small and the $N_l > 0$ are large parameters to be chosen later. **Notice**: the domain

$$D_l^a := \{(y_1, y_2) : \varepsilon_l y_1^{a_l} < |y_2| \le N_l y_1^{a_l}\}$$

corresponding to D_l in the adapted coordinates y is κ^l-homogeneous and contains the cluster of roots [l], while the domain E_l^a corresponding to E_l can be viewed as a domain of transition between two different homogeneities.

Clusters of roots

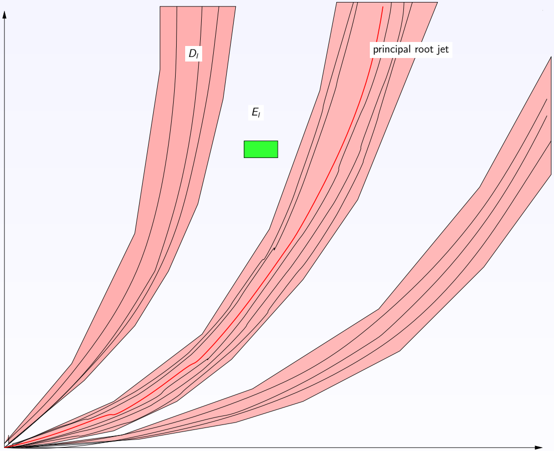

Figure: Clusters of roots

Contribution by D_l to $J(\xi)$

This can be treated somewhat similarly as the case of adapted coordinates: by using dyadic decompositions and subsequent re-scalings by means of the dilations δ_r^l associated to the weight κ^l, we may decompose into dyadic pieces $J_k(\xi)$, given by

$$2^{-k|\kappa^l|}\int_{\mathbb{R}_+^2} e^{-i\left(2^{-k}\xi_3\phi^k(y)+\xi_2\psi(2^{-k\kappa_1^l}y_1)+2^{-k\kappa_1^l}\xi_1 y_1+2^{-k\kappa_2^l}\xi_2 y_2\right)}\eta(\delta_{2^{-k}}^l y)\chi(y)\,dy,$$

where $\phi^k(y)=\phi^a_{\kappa^l}(y)+\text{error term}$.

Obstacle: since $1-m\kappa_1^l>\kappa_2^l-m\kappa_1^l>0$, the contribution of the non-linearity ψ to the complete phase of the corresponding oscillatory integrals may be large, compared to the term containing ϕ^k, so that we are only allowed to apply van der Corput's estimate to the integration with respect to the variable y_2 if we want to reduce to one-dimensional oscillatory integrals! This requires a good control on the multiplicities of roots of $\partial_2^2\phi^a_{\kappa^l}$ at points y^0 in the corresponding annulus $\mathcal{A}$ not lying on the y_1 axis (which corresponds to the principal root jet in the coordinates y).

Good news: these multiplicities are bounded by $d_h(\phi^a_{\kappa^l}) - 2$, where $d_h(\phi^a_{\kappa^l})$ denotes the homogeneous distance of $\phi^a_{\kappa^l}$, and it is evident from the geometry of the Newton polyhedron of ϕ^a that $d_h(\phi^a_{\kappa^l}) < d(\phi^a) = h$, so that for every point y^0 in $\mathcal{A} \cap D_l$ there is some $j \in \{2, \dots, h\}$ such that

$$\partial_2^j \phi^a_{\kappa^l}(y^0) \neq 0.$$

Contribution by E_l to $J(\xi)$

In E_l, we perform a separate dyadic decomposition in both variables y_1 and y_2, so that we geometrically decompose E_l into dyadic rectangles of size $2^{-j} \times 2^{-k}$, and then re-scale in both variables so that these rectangles become the standard cube, say, $[1,2] \times [1,2]$.

The phase functions $\phi^a_{j,k}$ that one obtains after these re-scalings satisfy the estimate

$$\partial_2^2 \phi^a_{j,k}(y^0) \neq 0 \quad \text{for every } y^0 \in [1,2] \times [1,2].$$

Since $h \geq 2$, this clearly suffices to obtain the necessary order of decay of the Fourier transform of these dyadic pieces. Moreover, scaling back to the original dyadic rectangles, a careful analysis of the dependency of the corresponding estimates on the parameters j, k shows that it is indeed possible to sum theses estimates and obtain the same type of estimate for the contributions by the domains E_l as for the domains D_l, even without logarithmic factor.

Step 3: Contribution by the domain D_λ containing the principal root jet

Note: So far, we have been able to reduce our estimations to van der Corput type lemmata, i.e., to one-dimensional oscillatory integrals!

In contrast, the study of the domain D_λ will require the estimation of genuinely 2-dimensional oscillatory integrals.

In the adapted coordinates y, the domain D_λ is given by $|y_2| \leq N_\lambda y_1^{a_\lambda}$. Cover it by a finite number of κ^λ-homogeneous subdomains of the form $|y_2 - cy_1^{a_\lambda}| \leq \varepsilon_0 y_1^{a_\lambda}$, where $c \in [-N_\lambda, N_\lambda]$, and where, for a given c, we may choose $\varepsilon_0 > 0$ suitably small.
Recalling that $\psi(x_1) = x_1^m \omega(x_1)$, with $\omega(0) \neq 0$, we can thus reduce to estimating oscillatory integrals

$$J^c(\xi) = \int_{\mathbb{R}^2_+} e^{iF(y,\xi)} \rho\Big(\frac{y_2 - cy_1^{a_\lambda}}{\varepsilon_0 x_1^{a_\lambda}}\Big) \eta(y)\, dy, \tag{1.10}$$

with a phase function

$$F(y,\xi) := \xi_3 \phi^a(y) + \xi_1 y_1 + \xi_2 y_1^m \omega(y_1) + \xi_2 y_2.$$

Arguing in a similar way as in the case of adapted coordinates, and recalling that $\phi^a_{\mathrm{pr}} = \phi^a_{\kappa^\lambda}$, we may again perform a dyadic decomposition and re-scale by means of the dilations δ^λ_r, in order to write

$$J^c(\xi) = \sum_{k=k_0}^{\infty} J_k(\xi),$$

where

$$J_k(\xi) = 2^{-|\kappa^\lambda|k} \int e^{i2^{-k}\xi_3 F_k(y,s)} \rho\Big(\frac{y_2 - c y_1^{a_\lambda}}{\varepsilon_0 y_1^{a_\lambda}}\Big) \eta(\delta^\lambda_{2^{-k}} y)\, \chi(y)\, dy, \qquad (1.11)$$

with

$$F_k(y,s) := \phi^a_{\mathrm{pr}}((y_1, y_2)) + s_1 y_1 + S_2 y_1^m \omega(2^{-\kappa^\lambda_1 k} y_1) + s_2 y_2 + \mathrm{error},$$

where $s := (s_1, s_2, S_2)$ is given by

$$s_1 := 2^{(1-\kappa^\lambda_1)k} \frac{\xi_1}{\xi_3},\ \ s_2 := 2^{(1-\kappa^\lambda_2)k} \frac{\xi_2}{\xi_3},\ \ S_2 := 2^{(\kappa^\lambda_2 - m\kappa^\lambda_1)k} s_2.$$

Note that $2 \leq m < a_\lambda = \kappa_2^\lambda / \kappa_1^\lambda$ and $k \gg 1$, so that $|S_2| \gg |s_2|$, and that here

$$y_1 \sim 1 \text{ and } |y_2 - c y_1^{a_\lambda}| \lesssim \varepsilon_0$$

Recall also that we are assuming that $|\xi| \sim |\xi_3|$.
One is thus led to the estimation of oscillatory integrals depending on certain parameters (here s_1, s_2, S_2) which may have various relative sizes.

- If $|S_2| \geq M$ for some $M \gg 1$, apply van der Corput's lemma to the y_1- integration, with $n = 2$.
- So, we may assume that $|S_2| < M$, so that in particular $|s_2| \ll 1$, and indeed that also $|s_1| < N$, if $N \gg M$.

We may reduce to the case where

$$\partial_2^j \phi^a_{\mathrm{pr}}(1, c) = \partial_2^j \phi^a_{\kappa^\lambda}(1, c) = 0 \text{ for } 1 \leq j < h, \tag{1.12}$$

for otherwise an integration by parts in y_2 (if $j = 1$) or a simple application of the van der Corput type lemma yields a suitable estimate as before.

The case where $c > 0$ can easily be reduced to the case $c = 0$ by performing another change of variables $y_2 \mapsto y_2 + c y_1^{a_\lambda}$ in the integral defining $J_k(\xi)$. I
Indeed, one can show that our assumption (1.12) implies that $a_\lambda = \kappa_2^\lambda / \kappa_1^\lambda \in \mathbb{N}$, and one checks that the new coordinates are again adapted to ϕ.

So, let us assume that $c = 0$. Then necessarily $\phi_{\mathrm{pr}}^a(1,0) \neq 0$, for otherwise ϕ_{pr}^a would have a root of multiplicity at least h at $(1,0)$, which would contradict our convention.
Assuming without loss of generality that $\phi_{\mathrm{pr}}^a(1,0) = 1$, we can write

$$\phi_{\mathrm{pr}}^a(y_1, y_2) = y_2^B Q(y_1, y_2) + y_1^n,$$

where Q is a κ^λ-homogeneous polynomial such that $Q(1,0) \neq 0$, and where $B \geq h > 2$.

Recall that we are assuming that (s_1, S_2) is from a compact set K . Thus it suffice to show that, given any point $(s_1^0, S_2^0) \in K$ and any point $y_1^0 \sim 1$, there exist a neighborhood U of (s_1^0, S_2^0), a neighborhood V of $(y_1^0, 0)$ and some $\sigma > 1/h$ so that

$$|J_k(\xi)| \lesssim \frac{2^{-k|\kappa|}}{(1 + 2^{-k}|\xi|)^\sigma} \tag{1.13}$$

for every $(s_1, S_2) \in U$, provided the function χ in the definition of $J_k(\xi)$ is supported in V, and ε_0 and k are chosen sufficiently small, respectively large. Summing over all k, this will clearly imply an estimate as in (1.1), even without logarithmic factor.

$F_k(y,s)$ can be viewed as a small C^∞- perturbation of the function

$$F_{\rm pr}(y) := y_2^B Q(y_1,y_2) + s_1^0 y_1 + S_2^0\omega(0) y_1^m + y_1^n.$$

Thus, if $\nabla F_{\rm pr}(y_1^0,0)\neq 0$, then we obtain (1.13), with $\sigma=1$, simply by integration by parts.
Assume next that $(y_1^0,0)$ is a critical point of $F_{\rm pr}$. Then y_1^0 is a critical point of the polynomial function

$$g(y_1) := s_1^0 y_1 + S_2^0\omega(0) y_1^m + y_1^n,$$

Note that $1\leq m<n$, since $n=1/\kappa_1^\lambda > \kappa_2^\lambda/\kappa_1^\lambda > m$. But then g'' and g''' cannot also vanish simultaneously at y_1^0 (" van der Monde det."), so that there are positive constants $c_1, c_2>0$ and a compact neighborhood V of y_1^0 such that

$$c_1 \leq \sum_{j=2}^{3} |g^{(j)}(y_1)| \leq c_2 \quad \text{for every } y_1\in V.$$

Since $y_2^0 = 0$, we may thus apply the van der Corput type lemma (if U, V are sufficiently small) and obtain estimate (1.13), with $\sigma = 1/3$, so that we are done provided $h > 3$. Notice also that if $g''(y_1^0) \neq 0$, then by the same type of argument we see that (1.13) holds true with $\sigma = 1/2 > 1/h$.

Assume finally that $2 < h \leq 3$, and that $g'(y_1^0) = g''(y_1^0) = 0$. Then

$$\frac{1}{\kappa_1^\lambda + \kappa_2^\lambda} = h \leq 3 \quad \text{and} \quad \frac{\kappa_2^\lambda}{\kappa_1^\lambda} > m \geq 2,$$

so that $1/\kappa_2^\lambda < 9/2$. Since $B \leq 1/\kappa_2^\lambda$ and $h \leq B < 9/2$, either $B = 4$ or $B = 3$. Translate the critical point $(y_1^0, 0)$ of F_{pr} to the origin by considering the function

$$F^\sharp_{\mathrm{pr}}(z) := F_{\mathrm{pr}}(y_1^0 + z_1, z_2) - g(y_1^0) = z_2^B Q(y_1^0 + z_1, z_2) + \frac{1}{6} g^{(3)}(y_1^0)\, z_1^3 + \dots .$$

Then $h^\sharp := h(F^\sharp) = \frac{1}{1/3 + 1/B} < 2.$ We may thus apply Duistermaat's results to this part of $J_k(\xi)$ and obtain estimate (1.13), with $\sigma = 1/h^\sharp > 1/h$. Note that Duistermaat's estimates are stable under small perturbations!

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