Aspects of harmonic analysis related to hypersurfaces, and Newton diagrams Part I

Detlef Müller
joint work with I. Ikromov

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References

- D. Müller: *Problems of Harmonic Analysis related to finite type hypersurfaces in $\mathbb{R}^3$, and Newton polyhedra*; to appear in the Proceedings of the Conference in Honor of E.M. Stein.; http://www.math.uni-kiel.de/analysis/mueller/pub.shtml
- I.A. Ikromov, M. Kempe and D. Müller: *Estimates for maximal functions associated to hypersurfaces in $\mathbb{R}^3$ and related problems of harmonic analysis* , Acta Math. 204 (2010), 151–271.
- I. Ikromov, D. Müller: *L^p-L^2 Fourier restriction for hypersurfaces in $\mathbb{R}^3$, I, II*; in preparation.
- I. Ikromov, D. Müller: *Uniform estimates for the Fourier transform of surface carried measures on hypersurfaces in $\mathbb{R}^3$ and an application to Fourier restriction*, J. Fourier Anal. Appl. 17 (2011), 1292–1332.
- I. Ikromov, D. Müller: *On adapted coordinate systems*, Trans. Amer. Math. Soc. 363 (2011), 2821–2848.

1. Introduction: Two (to three) interrelated problems

$S =$ smooth, finite type hypersurface in $\mathbb{R}^3$,

$d\mu := \rho d\sigma$, $d\sigma :=$ surface measure on S, $0 \le \rho \in C_0^\infty(S)$

A

Sharp uniform decay estimates for $\widehat{d\mu}(\xi) := \int_S e^{-i\xi x} d\mu(x)$, $\xi \in \mathbb{R}^3$?

B

For which p's do we have a Fourier restriction estimate

$$\Big(\int_S |\hat{f}(x)|^2 \, d\mu(x)\Big)^{1/2} \le C\|f\|_{L^p(\mathbb{R}^3)}, \quad f \in \mathcal{S}(\mathbb{R}^3) \ ?$$

C

$L^p(\mathbb{R}^3)$ - boundedness of the maximal operator $\mathcal{M}f(x) := \sup_{t>0} |A_t f(x)|$, where $A_t f(x) := \int_S f(x - ty) d\mu(y)$.

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Short History of these problems

(A) Estimation of oscillatory integrals:

B. Riemann (1854):	appear implicitly in his work
Best understood:	*Convex hypersurfaces of finite line type:*
	B. Randol (1969)
	I. Svensson(1971) H. Schulz (1991)
	J. Bruna, A. Nagel, S. Wainger (1988)
Non-convex case:	
A.N. Varchenko (1976) :	$\int e^{i\lambda\phi(x_1,x_2)}a(x_1,x_2)\,dx$, ϕ analytic
V.N. Karpushkin (1984):	$\int e^{i\lambda(\phi(x_1,x_2)+r(x_1,x_2))}a(x_1,x_2)\,dx$, ϕ analytic

(B) The Fourier-restriction problem: E.M. Stein (1967).
E.M. Stein and P.A. Tomas (1975) :

$$\Big(\int_{S^{n-1}} |\hat{f}(x)|^2\,d\mu(x)\Big)^{1/2} \leq C\|f\|_{L^p(\mathbb{R}^n)}$$

iff $p' \geq 2(\frac{2}{n-1}+1)$.

Representation of S as a graph of ϕ

$S \subset \mathbb{R}^3$ smooth, finite type hypersurface; $x^0 \in S$:

By localization near x^0 and application of Euclidean motion of $\mathbb{R}^3$ we may assume: $x^0 = (0,0,0)$, and

$$S = \{(x_1, x_2, \phi(x_1, x_2)) : (x_1, x_2) \in \Omega\},$$

where $\phi \in C^\infty(\Omega)$ s.t. $\phi(0,0) = 0$, $\nabla\phi(0,0) = 0$. If

$$\phi(x_1, x_2) \sim \sum_{j,k=0}^{\infty} c_{jk} x_1^j x_2^k$$

is the Taylor series of ϕ, define the Taylor support of ϕ at $(0,0)$ by

$$\mathcal{T}(\phi) := \{(j,k) \in \mathbb{N}^2 : c_{jk} \neq 0\}.$$

NOTICE: $\mathcal{T}(\phi) \neq \emptyset$, since ϕ is of finite type at the origin!

2. Newton polyhedra, and adapted coordinates

1. Newton polyhedron:

$$\mathcal{N}(\phi) := \operatorname{conv} \bigcup_{(j,k)\in\mathcal{T}(\phi)} (j,k) + \mathbb{R}_+^2$$

   Newton diagram $\mathcal{N}_d(\phi)$: Union of all compact faces of $\mathcal{N}(\phi)$

2. Newton distance : $d = d(\phi)$ is given by the coordinate d of the point (d, d) at which the bisectrix $t_1 = t_2$ intersects the boundary of the Newton polyhedron.

3. Principal face $\pi(\phi)$: The face of minimal dimension containing the point (d, d).

4. Principal part of ϕ :

$$\phi_{\mathrm{pr}}(x_1, x_2) := \sum_{(j,k)\in\pi(\phi)} c_{jk} x_1^j x_2^k$$

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Figure 1

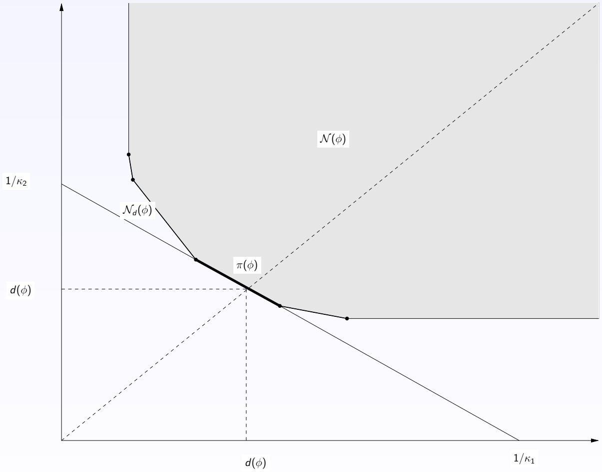

Figure: Newton polyhedron

Adapted coordinates

Height of ϕ :

$$h(\phi) := \sup\{d_y\},$$

where the supremum is taken over all local analytic (resp. smooth) coordinate systems $y = (y_1, y_2)$ at the origin, and where d_y is the Newton distance of ϕ when expressed in the coordinates x.

NOTICE: The height is invariant under local smooth changes of coordinates at the origin!

A coordinate system x is said to be adapted to ϕ if $h(\phi) = d_x$.

Example 1. Let

$$\phi(x_1, x_2) := (x_2 - x_1^m)^n + x_1^\ell.$$

If $\ell > mn$, the coordinates are not adapted. Adapted coordinates are then $y_1 := x_1, y_2 := x_2 - x_1^m$, in which ϕ is given by

$$\phi^a(y) = y_2^n + y_1^\ell.$$

Example 1

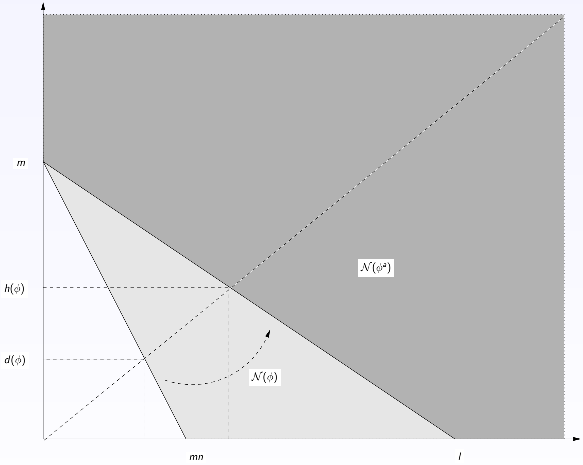

Figure: $\phi(x_1, x_2) := (x_2 - x_1^m)^n + x_1^\ell \quad (\ell > mn)$

Supporting lines and homogeneities

Let $\kappa = (\kappa_1, \kappa_2)$ with, say, $\kappa_2 \geq \kappa_1 > 0$, be a given weight, with corresponding dilations

$$\delta_r(x_1, x_2) := (r^{\kappa_1} x_1, r^{\kappa_2} x_2), \quad r > 0.$$

F on $\mathbb{R}^2$ is κ-homogeneous of degree a, (short: mixed homogeneous) if

$$F(\delta_r x) = r^a F(x) \quad \forall r > 0, x \in \mathbb{R}^2.$$

Assume that $L_\kappa := \{(t_1, t_2) \in \mathbb{R}^2 : \kappa_1 t_1 + \kappa_2 t_2 = a\}$ is a supporting line to the Newton polyhedron $\mathcal{N}(\phi)$ of ϕ. The κ-principal part of ϕ

$$\phi_\kappa(x_1, x_2) := \sum_{(j,k) \in L_\kappa} c_{jk} x_1^j x_2^k$$

is κ-homogeneous of degree a.

$$\phi(x_1, x_2) = \phi_\kappa(x_1, x_2) + \text{ terms of higher } \kappa\text{-degree.}$$

Principal weight

Assume $\pi(\phi)$ is a compact edge; then it lies on a unique principal line

$$L := \{(t_1, t_2) \in \mathbb{R}^2 : \kappa_1 t_1 + \kappa_2 t_2 = 1\},$$

with $\kappa_1, \kappa_2 > 0$. We may assume that $\kappa_1 \leq \kappa_2$ (possibly after permutation of coordinates). The weight $\kappa = (\kappa_1, \kappa_2)$ will be called the principal weight associated to ϕ. Then $\phi_{\mathrm{pr}} = \phi_\kappa$, and

$$d = \frac{1}{\kappa_1 + \kappa_2} = \frac{1}{|\kappa|}, \qquad (3.1)$$

$$\phi_\kappa(x_1, x_2) = c x_1^{\nu_1} x_2^{\nu_2} \prod_{l=1}^{M} (x_2^q - \lambda_l x_1^p)^{n_l}, \qquad (3.2)$$

with $M \geq 1$, distinct non-trivial "roots" $\lambda_l \in \mathbb{C} \setminus \{0\}$ of multiplicities $n_l \in \mathbb{N} \setminus \{0\}$, and trivial roots of multiplicities $\nu_1, \nu_2 \in \mathbb{N}$ at the coordinate axes. Here, p and q have no common divisor, and $\kappa_2/\kappa_1 = p/q$.

Conditions for Adaptedness

Let $P \in \mathbb{R}[x_1, x_2]$ be a κ- homogeneous polynomial with $\nabla P(0,0) = 0$, let

$$m(P) := \mathrm{ord}_{S^1} P$$

be the maximal order of vanishing of P along the unit circle S^1 centered at the origin.

The homogeneous distance of a κ-homogeneous polynomial P (such as $P = \phi_{\mathrm{pr}}$) is given by

$$d_h(P) := 1/(\kappa_1 + \kappa_2) = 1/|\kappa|.$$

Notice that $(d_h(P), d_h(P))$ is just the point of intersection of the line given by $\kappa_1 t_1 + \kappa_2 t_2 = 1$ with the bi-sectrix $t_1 = t_2$. The height of P can then be computed by means of the formula

$$h(P) = \max\{m(P), d_h(P)\}. \tag{3.3}$$

Theorem

The coordinates x are adapted to ϕ if and only if one of the following conditions is satisfied:

(a) *The principal face $\pi(\phi)$ of the Newton polyhedron is a compact edge, and $m(\phi_{\mathrm{pr}}) \leq d(\phi)$.*

(b) *$\pi(\phi)$ is a vertex.*

(c) *$\pi(\phi)$ is an unbounded edge.*

It can be shown that (a) applies whenever $\pi(\phi)$ is a compact edge and $\kappa_2/\kappa_1 \notin \mathbb{N}$; in this case we even have $m(\phi_{\mathrm{pr}}) < d(\phi)$

Theorem (Varchenko; Phong, J. Sturm, Stein (analytic ϕ); I.,M.)

There always exist adapted smooth coordinates y, of the form $y_1 = x_1,\ y_2 = x_2 - \psi(x_1)$.

Construction of adapted coordinates

Assume the coordinates (x_1, x_2) are not adapted to ϕ. $\Longrightarrow$ $\pi(\phi)$ is compact edge, $m := \kappa_2/\kappa_1 \in \mathbb{N}$, $p = m, q = 1$ in (3.2), and $m(\phi_{\mathrm{pr}}) > d(\phi)$.
$\Longrightarrow$ there is at least one, non-trivial real root $x_2 = \lambda_l x_1$ of ϕ_{pr} of multiplicity $n_l = m(\phi_{\mathrm{pr}}) > d(\phi)$. This root is unique. Putting $b_1 := \lambda_l$, we shall denote the corresponding root $x_2 = b_1 x_1$ of ϕ_{pr} as its principal root. Changing coordinates

$$y_1 := x_1,\ y_2 := x_2 - b_1 x_1^m,$$

we arrive at a "better" coordinate system $y = (y_1, y_2)$. Indeed, this change of coordinates will transform ϕ_{pr} into a function $\widetilde{\phi_{\mathrm{pr}}}$, where the principal face of $\widetilde{\phi_{\mathrm{pr}}}$ will be a horizontal half-line at level $t_2 = m(\phi_{\mathrm{pr}})$, so that $d(\widetilde{\phi_{\mathrm{pr}}}) > d(\phi)$, and correspondingly one finds that $d(\tilde{\phi}) > d(\phi)$, if $\tilde{\phi}$ expresses ϕ is the coordinates y.

Essentially by iterating this procedure, we arrive at Varchenko's algorithm for the construction of an adapted coordinate system.

In conclusion: there exists a smooth real-valued function ψ (which we may choose as the so-called principal root jet of ϕ) of the form

$$\psi(x_1) = x_1^m \omega(x_1) \tag{3.4}$$

with $\omega(0) \neq 0$, defined on a neighborhood of the origin such that an adapted coordinate system (y_1, y_2) for ϕ is given locally near the origin by means of the (in general non-linear) shear

$$y_1 := x_1, \;\; y_2 := x_2 - \psi(x_1). \tag{3.5}$$

In these adapted coordinates, ϕ is given by

$$\phi^a(y) := \phi(y_1, y_2 + \psi(y_1)). \tag{3.6}$$

Example 1. $\phi(x_1, x_2) := (x_2 - x_1^m)^n + x_1^\ell, \;\ell > mn$. The coordinates x are not adapted. Indeed, $\phi_{\rm pr}(x_1, x_2) = (x_2 - x_1^m)^n$, $d(\phi) = 1/(1/n + 1/(mn)) = mn/(m+1)$ and $m(\phi_{\rm pr}) = n > d(\phi)$. Adapted coordinates are given by $y_1 := x_1, y_2 := x_2 - x_1^m$, in which ϕ is expressed by $\phi^a(y) = y_2^n + y_1^\ell$.

Linearly adapted coordinates

If $m = \kappa_2/\kappa_1 = 1$ in the first step of Varchenko's algorithm, then a linear change of coordinates of the form $y_1 = x_1, y_2 = x_2 - b_1 x_1$ will transform ϕ into a function $\tilde{\phi}$. Since all of our problems A - C are invariant under such linear changes of coordinates, by replacing our original coordinates (x_1, x_2) by (y_1, y_2) and ϕ by $\tilde{\phi}$, we may in the sequel always assume the following

CONVENTION:
– either our coordinates (x_1, x_2) are adapted, or
– they are not adapted and

$$m = \kappa_2/\kappa_1 \qquad \text{is an integer} \quad \geq 2. \tag{3.7}$$

A linear, non-adapted coordinate system for which (3.7) holds true will be called linearly adapted to ϕ.

A. Decay of the Fourier transform of the surface measure

Write $\widehat{\mu}(\xi)$ as an oscillatory integral

$$\widehat{\mu}(\xi) =: J(\xi) = \int_\Omega e^{-i(\xi_3\phi(x_1,x_2)+\xi_1x_1+\xi_2x_2)}\eta(x)\,dx, \quad \xi \in \mathbb{R}^3,$$

$\eta \in C_0^\infty(\Omega)$. Since $\nabla\phi(0,0) = 0$, the complete phase in this oscillatory integral will have no critical point on the support of η unless $|\xi_1| + |\xi_2| \ll |\xi_3|$, provided Ω is chosen sufficiently small. Integrations by parts then show that $\widehat{\mu}(\xi) = O(|\xi|^{-N})$ as $|\xi| \to \infty$, for every $N \in \mathbb{N}$, unless $|\xi_1| + |\xi_2| \ll |\xi_3|$.

We may thus focus on the latter case. In this case, by writing $\lambda = -\xi_3$ and $\xi_j = s_j\lambda,\ j = 1, 2$, we are reduced to estimating two-dimensional oscillatory integrals of the form

$$I(\lambda; s) := \int e^{i\lambda(\phi(x_1,x_2)+s_1x_1+s_2x_2)}\eta(x_1, x_2)\,dx_1\,dx_2,$$

where $\lambda \gg 1$, and that $s = (s_1, s_2) \in \mathbb{R}^2$ are sufficiently small parameters, provided that η is supported in a sufficiently small neighborhood of the origin. The phase is a linear perturbation of ϕ!

Theorem (Bernstein-Gelfand; Atiyah)

If ϕ is analytic (on $\mathbb{R}^n$), then

$$\int e^{i\lambda\phi(x)}\eta(x)\,dx \sim \sum_{k=0}^{\infty}\sum_{j=0}^{n-1} a_{j,k}(\phi)\lambda^{-r_k}\log(\lambda)^j, \tag{4.1}$$

provided the support of η is sufficiently small.

Here, the r_k form an increasing sequence of rational numbers consisting of a finite number of arithmetic progressions, which depends only on the zero set of ϕ, and the $a_{j,k}$ are distributions with respect to the cut-off function η. The proof is based on Hironaka's theorem.

Varchenko's exponent $\nu(\phi) \in \{0, 1\}$ for $n = 2$: If there exists an adapted local coordinate system y near the origin such that the principal face $\pi(\phi^a)$ of ϕ^a, is a vertex, and if $h(\phi) \geq 2$, then we put $\nu(\phi) := 1$; otherwise, we put $\nu(\phi) := 0$.

Remark: the first condition is equivalent to the following one:
If y is any adapted local coordinate system at the origin, then either $\pi(\phi^a)$ is a vertex, or a compact edge and $m(\phi^a_{\mathrm{pr}}) = d(\phi^a)$.

Varchenko: the leading exponent in (4.1) is given by $r_0 = 1/h(\phi)$, and $\nu(\phi)$ is the maximal j for which $a_{j,k}(\phi) \neq 0$. This implies in particular that

$$|I(\lambda;0)| \leq C\lambda^{-\frac{1}{h(\phi)}} \log(\lambda)^{\nu(\phi)}, \quad \lambda \gg 1, \tag{4.2}$$

and this estimate is sharp in the exponents.
Karpushkin: this estimate is stable under sufficiently small analytic perturbations of ϕ (analogous results are known to be wrong in higher dimensions!).

In particular, we obtain the following uniform estimate for $\hat{\mu}$,

$$|\widehat{\mu}(\xi)| \le C(1+|\xi|)^{-\frac{1}{h(\phi)}} \log(2+|\xi|)^{\nu(\phi)}, \quad \xi \in \mathbb{R}^3, \tag{4.3}$$

Theorem (Ikromov, M.)

Let $S = \mathrm{graph}(\phi)$, ϕ smooth and finite type. Then there exists a neighborhood $U \subset S$ of $x^0 = 0$ such that for every $\rho \in C_0^\infty(U)$ the following estimate holds true for every $\xi \in \mathbb{R}^3$:

$$|\widehat{d\mu}(\xi)| \le C\,\|\rho\|_{C^3(S)}\,(\log(2+|\xi|))^{\nu(\phi)}(1+|\xi|)^{-1/h(\phi)} \tag{4.4}$$

Remark: For ϕ smooth, M. Greenblatt had obtained such estimates for ξ normal to S at 0.

Sharpness

Let N be a unit normal to S at $x^0 = 0$, and put

$$J(\lambda) := \widehat{d\mu}(\lambda N) = \iint e^{\pm i\lambda\phi(x_1,x_2)} a(x_1,x_2)\, dx_1 dx_2, \quad \lambda > 0.$$

Proposition

If in an adapted coordinates system the principal face $\pi(\phi^a)$ is a compact set (i.e. a compact edge or a vertex), then the following limit

$$\lim_{\lambda\to+\infty} \frac{\lambda^{1/h(\phi)}}{\log \lambda^{\nu(\Phi)}} J(\lambda) = C \cdot a(0,0),$$

exists, where C is a non-zero constant depending on ϕ only.

Remarks:

1. This improves on a result by M. Greenblatt, who proved that this limit exists for some sequence of $\lambda_k \to \infty$.
2. If the principal face $\pi(\phi^a)$ is unbounded, then the estimate in the theorem may fail to be sharp, if ϕ is non-analytic, as the following example by A. Iosevich and E. Sawyer shows: If

$$\Phi(x_1, x_2) := x_2^2 + e^{-1/|x_1|^\alpha},$$

then

$$|J(\lambda)| \asymp \frac{1}{\lambda^{1/2} \log \lambda^{1/\alpha}} \quad \text{as} \quad \lambda \to +\infty.$$

Here, $\nu(\phi) = 0$.

SHORT BREAK (CHANGE OF FILE)!

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