# An $H_1$-BMO duality for Markov Semigroups of Operators

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# Question

$(\mathcal{M}, \mu)$: sigma finite measure space;
$L_p(\mathcal{M})$: $L_p$ spaces, i.e.

$$L_p(\mathcal{M}) = \{f \in L_0(\mathcal{M}), \|f\|_p = (\int |f|^p d\mu)^{\frac{1}{p}} < \infty\}.$$

$(T_t)_t$: "nice" $c_0$ semigroup of operators;
Littlewood-Paley theory:

$$\|(\int_0^\infty |\frac{\partial T_t f}{\partial t}|^2 t dt)^{\frac{1}{2}}\|_{L_p} \simeq \|f\|_{L_p^0},$$

$1 < p < \infty$. No "direct" use of local properties of $(\mathcal{M}, \mu)$.

Semigroup-analogues of classical (real) BMO and $H_1$ spaces theory

$$(H_1)^* = BMO, \quad [BMO, H_1]_{\frac{1}{p}} = L_p(\mathcal{M}), 1 < p < \infty?$$

Work on martingales: Burkholder-Gundy, etc.; Pisier/Xu, Junge/Musat, Parcet, Perrin, etc.

# BMO and Hardy spaces on $\mathbb{R}^n$

$(\mathcal{M},\tau)=(L_\infty(\mathbb{R}),dx)$.
$f\in L_1^{loc}(\mathbb{R})$
$f_I=\frac{1}{|I|}\int_I fdx$.

$$\|f\|^2_{\mathrm{BMO}(\mathbb{R})}=\sup_{I\subset\mathbb{R}}\frac{1}{|I|}\int_I|f(x)-f_I|^2dx,=\sup_{I\subset\mathbb{R}}\{(|f|^2)_I-|f_I|^2\}.$$

$\mathrm{BMO}(\mathbb{R})=\{f\in L_1(\mathbb{R},\frac{dx}{1+x^2});\|f\|_{BMO(\mathbb{R})}<\infty\}$.

$$\begin{aligned}G(f)(x)&=(\int_0^\infty|\nabla P_t(f)|^2tdt)^{\frac12},\\ S(f)(x)&=(\int_{\{(y,t):|y-x|<t\}}|\nabla P_t(f)|^2tdy\frac{dt}{t^n})^{\frac12}.\\ \|f\|_{H_1}&=\|G(f)\|_{L_1}+\|f\|_{L_1}\simeq\|S(f)\|_{L_1}+\|f\|_{L_1}.\\ H_1(\mathbb{R})&=\{f\in L_1(\mathbb{R});\|f\|_{H_1}<\infty\}.\end{aligned}$$

Garnett, Koosis, Varopoulos,... Replacing $\frac{1}{|I|}\int_I\cdot$ by $T_t$,
$\|f\|^2_{BMO(\mathbb{R})}\simeq\sup_t\|T_t|f-T_tf(\cdot)|^2\|_\infty=\sup_t\|T_t|f|^2-|T_tf|^2\|_\infty$.

# BMO and Hardy spaces on metric spaces

$(M, \mu)$= metric space with a doubling measure.
$(T_t)_t$= semigroups of operators with fast decreasing kernels

$$\|f\|^2_{\mathrm{BMO(M)}} = \sup_{\mathrm{ball} I \subset M} \frac{1}{|I|} \int_I |f - T_{r_I^m} f|^2 dx.$$

$$S(f)(x) = (\int_{\{(y,t):|y-x|<t^{\frac{1}{m}}\}} |\partial_t T_t(f)|^2 t d\mu(y) \frac{dt}{V(x, t^{\frac{1}{m}})})^{\frac{1}{2}}.$$

$$\|f\|_{H_1(M)} = \|S(f)\|_{L_1} + \|f\|_{L_1}.$$

$H_1(M)$= $\{f \in L_1(M); \|f\|_{H_1} < \infty\}$.

( Auscher-Duong-Mcintoch, 2004; Duong-Yan, J. AMS 2005; Hofmann-Lu-Mitrea-Mitrea-Yan, Mem. AMS 2012; etc.)

$$(H_1(M))^* = BMO(M), [BMO(M), H_1(M)]_{\frac{1}{p}} = L_p(M).$$

Non-doubling measure: Mauceri-Meda, Portal, Garcia-Cuerva-Torrea,....Tolsa,...

# Example of Noncommutative $L_p$ spaces

$G$: (nonabelian) discrete group. e.g. $G = \mathbb{Z}, \mathbb{F}_2$,
$\delta_g, g \in G$: the canonical basis of $\ell_2(G)$.

$\lambda_g$: left regular representation of $G$ on $\ell_2(G)$,

$$\lambda_g(\delta_h) = \delta_{gh}, \quad \text{for } g, h \in G.$$

$L_\infty(\hat{\mathbb{G}})$: the $w^*$ closure of $Span\{\lambda_g\}$'s in $B(\ell_2(G))$.
Example: $G = \mathbb{Z}, \lambda_k = e^{ik\theta}, L_\infty(\hat{\mathbb{G}}) = L_\infty(\mathbb{T})$.
$\tau$: For $f = \sum_g f_g \lambda_g$,

$$\tau f = f_e.$$

$$||f||_{L^p(\widehat{\mathbb{G}})} = [\tau(|f|^p)]^{\frac{1}{p}}, 1 \le p < \infty.$$

Example: $G = \mathbb{Z}, \tau f = \hat{f}(0) = \int f, L_p(\hat{\mathbb{G}}) = L_p(\mathbb{T})$.
When $G$ is abelian, $\tau = \int_{\hat{G}} d\mu$. $L^p(\widehat{\mathbb{G}}) = L^p(\widehat{G})$.

Problem Find (nontrivial) $L^p(\widehat{\mathbb{G}})$ bounded multipliers

$$M_m : \lambda_k \to m(k)\lambda_k.$$

# Markov Semigroups of Operators

$(\mathcal{M}, \mu)$: Sigma finite measure space,
$(T_t)_{t\geq 0}$: a semigroup of operators on $L_2(\mathcal{M})$,
We say $(T_t)_t$ is Markov, if

- $T_t$ are normal contractions on $L_\infty(\mathcal{M}, \mu)$.
- $T_t$ are symmetric i.e. $\langle T_t f, g\rangle = \langle f, T_t g\rangle$ for $f, g \in L^1(\mathcal{M}) \cap L_\infty(\mathcal{M})$.
- $T_t(1) = 1$
- $T_t(f) \to f$ in the $w^*$ topology for $f \in \mathcal{M}$.

Infinitesimal generator: $L = -\frac{\partial T_t}{\partial t}|_{t=0}$; $T_t = e^{-tL}$.

Abstract theories by E. Stein, Cowling, Mcintoch...., Junge/Xu, Le Merdy-Junge-Xu.

# BMO associated with $(T_t)_t$

$(\mathcal{M}, \mu)$: finite measure space;
$(T_t)_t$: Markov semigroup of operators
For $f \in L^2(\mathcal{M})$,

$$\|f\|_{BMO(T)} = \sup_t \|T_t|f - T_t f|^2\|_\infty^{\frac{1}{2}}.$$

$$\|f\|_{bmo(T)} = \sup_t \|T_t|f|^2 - |T_t f|^2\|_\infty^{\frac{1}{2}}.$$

$$(Property)\ \|f\|_{BMO(T)} \simeq \|f\|_{bmo(T)} + \sup_t \|T_t f - T_{2t} f\|_\infty \ \text{if } \Gamma_2 \geq 0;$$

$$BMO(T) = \{f \in L^2(\mathcal{M}); \|f\|_{BMO(T)} < \infty\}.$$

( Junge-M. Math. Ann. 2012; Junge-M-Parcet)

$$[BMO(T), L_1^0(\mathcal{M})]_{\frac{1}{p}} = L_p^0(\mathcal{M}).$$

$$\|L^{iu}\|_{L_\infty(\mathcal{M}) \to BMO(P)} < c.$$

Boundedness of fourier multipliers from $L_\infty(\mathcal{M})$ to $BMO$.
von Neumann algebras; $\|L^{iu}\|_{B(L_p)} \lesssim \max\{p, \frac{1}{p-1}\} u^{-|\frac{1}{2} - \frac{1}{p}|} e^{|\frac{\pi u}{2} - \frac{\pi u}{p}|}$.

# Carré du Champ, $\Gamma(\cdot,\cdot) = |\nabla \cdot|^2$

$(\mathcal{M}, d\mu) = (\mathbb{R}, dx)$,
$L = \Delta = \partial^2 x$;

$$2\partial f^* \partial g = \triangle(f^* g) - \triangle f^* g - f^* \triangle g.$$

For $T_t = e^{-tL}$, set

$$2\Gamma(f,g) = -L(f^* g) + L(f^*)g + f^* L(g);$$
$$2\Gamma_2(f,g) = -L(\Gamma(f,g)) + \Gamma(L(f),g) + \Gamma(f, L(g)).$$

analogues of $\partial f^* \partial g, \partial^2 f^* \partial^2 g$

$\Gamma(f,f) \geq 0$ iff $|T_t f|^2 \leq T_t |f|^2$.

We say $T_t$ satisfies $\Gamma_2 \geq 0$ if $\Gamma_2(f,f) \geq 0$.

P. A. Meyer, D. Bakry, M. Emery, X. D. Li, F. Baudoin-N. Garofalo, etc.
($\Gamma_2 \geq 0 \Leftrightarrow CD(0,\infty)$ criterion $\Leftrightarrow 2T_t|T_t f|^2 \leq T_{2t}|f|^2 + |T_{2t} f|^2$.)

Junge-M-Parcet; Quantumn metric spaces;..

## $H^1$ space associated with $(T_t)_t$

Recall we replace $\frac{1}{|I|}\int_I \cdot$ by $T_t$,
Replace $\int_{\{(y,t),|y-x|<t\}} \cdot dy \frac{dt}{t^n} = \int_0^\infty \int_{B_x(t)} \cdot dy \frac{dt}{t^n}$ by $\int_0^\infty T_t \cdot dt$
Replace $|\nabla \cdot|^2$ by $\Gamma(\cdot,\cdot)$.
$f \in L_1(\mathcal{M})$.

$$S(f) = (\int_0^\infty T_t \Gamma(T_t f, T_t f) dt)^{\frac{1}{2}}$$

$$G(f) = (\int_0^\infty \Gamma(T_t f, T_t f) dt)^{\frac{1}{2}}.$$

Let

$$\|f\|_{H_1^S(T)} = \|S(f)\|_{L_1(\mathcal{M})} + \|f\|_{L_1(\mathcal{M})}.$$

Example $\mathcal{M} = L^\infty(\mathbb{R})$, $T_t(f) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{|x|^2}{4t}} * f$.
$\Gamma(f,f) = |\nabla f|^2$, $H_1^S(T) = H_1(\mathbb{R})$.
Only show the good side, difficulties: missing of geometric tools

# Main results

Theorem 1 (M 2012) Let $(T_t)$ be a Markov semigroup of operators satisfying $\Gamma_2 \geq 0$. Then

$$BMO(T) \subset (H_1^S(T))^*$$

i.e.

$$|\langle f, g \rangle| \leq c \|f\|_{H_1^S(T)} \|g\|_{BMO(T)},$$

for nice $f, g$.

Examples:

$(T_t)_t = e^{t\triangle}$: Heat semigroups generated by the Laplace-Beltrami operator on a complete Riemannian manifold with nonnegative Ricci curvature;

$(T_t)_t = e^{t(\triangle - \nabla\phi\cdot\nabla)}$:Ornstein-Uhlenbeck semigroups on complete Riemannian manifold with the measure $e^{-\phi}d\mu$ satisfying Ricci$+\nabla^2\phi \geq 0$;

$(T_t)_t =$Markov semigroups of operators on group von Neumann algebras;

# Example: Free group

$\mathbb{F}_n$:=nonabelian group generated by $n$ generators $g_1, g_2, ..g_n$
i.e.
$\{g = g_{n_1}^{r_1} g_{n_2}^{r_2} \cdots g_{n_k}^{r_k}; 1 \leq n_1, n_2, n_k \leq n, n_i \neq n_{i+1}, r_1, r_2, ..r_k \in \mathbb{Z}\}$.
$e$: unit element.
$n = 1$: $\mathbb{F}_1 = \mathbb{Z}, e = 0$.
$\lambda_g$: $(\delta_h) \to \delta_{gh}$.
For $f = \sum_g f_g \lambda_g$,

$$\tau f = f_e.$$

$$||f||_{L^p(\widehat{\mathbb{F}}_n)} = [\tau(|f|^p)]^{\frac{1}{p}}, 1 \leq p < \infty.$$

$|g|$:=$|r_1| + |r_2| + \cdots + |r_k|$.
$T_t$: $\lambda_g \to e^{-t|g|}\lambda_g$ is a Markov semigroup of operators (Haagerup, Invent. 1979) + $\Gamma_2 \geq 0$ autuomatically.

Theorem 1 holds .

Question For $1 < p < \infty$, is $\lambda_g \to |g|^i \lambda_g$ bounded on $L^p(\widehat{\mathbb{F}}_n)$?
$n > 1$? Yes, and bounded from $H_1^S(T)$ to $L_1(\widehat{\mathbb{F}}_n)$.

# Main results

Theorem 2 (M 2012) Assume, in addition, that there exist constants $c_1, c_2, r > 0$ such that
(i) $\|T_t f - T_{t+\varepsilon t}\|_1 \leq c_1 \varepsilon^r \|f\|_1$, for all $t, \varepsilon > 0$ and $f \in L_1(\mathcal{M})$.
(ii) Let $M_t = \frac{1}{t}\int_0^t T_s ds$

$$\|(M_{8t}|T_t f|^2)^{\frac{1}{2}}\|_{L_1(\mathcal{M})} \leq c_2 \|f\|_{L_1(\mathcal{M})},$$

for all $t > 0$ and $f \in L_1^+(\mathcal{M})$. Then

- $(H_1^S(T))^* = BMO(T)$.
- $[BMO(T), H_1^S(T)]_{\frac{1}{p}} = L_p(\mathcal{M})$.
- $bmo(T) = BMO(T)$.
- $\|G(f)\|_{L_1(\mathcal{M})} \simeq \|S(f)\|_{L_1(\mathcal{M})}$.

Analyticity of $(T_t)_t$ on $L_1$ implies (i) with $r = 1$.

Gaussian upper bound of the kernel of $(T_t)_t$ implies (ii).

$(T_t)_t =$ Heat semigroups generated by the Laplace-Beltrami operator on a complete Riemannian manifold with nonnegative Ricci curvature;

# How about using $\partial_t$? Carleson Measure Estimate

View $T_s \int_0^s f_t dt$ as alternatives to $\frac{1}{|I|}\int_{\{I\times(0,|I|)\}} f_t dt d\mu$.
Call $\nu_t$ a Carleson measure on $\mathcal{M}\times(0,\infty)$, if

$$\| T_s \int_0^s d\nu_t \| < c.$$

Proof of Theorem 1 yields

$$|\langle f, g\rangle| \leq c\|(\int_0^\infty T_t|\partial_t T_t f|^2 t dt)^{\frac{1}{2}}\|_{L_1} \sup_t \| T_s \int_0^s |\partial_t T_t g|^2 t dt\|_{L_\infty}^{\frac{1}{2}},$$

Question

$$\sup_s \| T_s \int_0^s |\partial_t T_t g|^2 t dt\|_{L_\infty} \leq c\|g\|_{BMO(T)}^2?$$

Yes! for $P_t = e^{-t\sqrt{L}}$ subordinated semigroups (M. JFA 2008)
using $P_t f \leq \frac{t}{s} P_s f$ for $f \geq 0, t \geq s$.
In general,

$$\sup_s \|(\frac{1}{s}\int_s^{4s} T_t dt)\int_0^s \Gamma(T_t g, T_t g) dt\|_{L_\infty} \leq c\|g\|_{BMO(T)}^2.$$

# Examples on discrete groups

Consider $\phi$ a real valued function on a discrete group $G$. Define

$$\begin{aligned} L(\lambda_g) &= \phi(g)\lambda_g, \\ T_t(\lambda_g) &= e^{-t\phi(g)}\lambda_g. \end{aligned}$$

Then $(T_t)_t$ is a Markov semigroup of operators on $L_\infty(\hat{\mathbb{G}})$ if
$\phi(1) = 0, \phi(g) = \phi(g^{-1})$
and $\phi$ is conditionally negative,

$$\sum_{g,h} \overline{a_g} a_h \phi(g^{-1}h) \leq 0$$

for any complex numbers $a_g$ with $\sum_g a_g = 0$.
It is easy to compute by the definition that

$$\begin{aligned} \Gamma(\sum_g a_g \lambda_g) &= \sum_{g,h} \bar{a}_g a_h K_\phi(g,h)\lambda_{g^{-1}h}, \\ \Gamma_2(\sum_g a_g \lambda_g) &= \sum_{g,h} \bar{a}_g a_h K_\phi^2(g,h)\lambda_{g^{-1}h}, \end{aligned}$$

with $K_\phi(g,h) = \frac{-\phi(gh^{-1})+\phi(g)+\phi(h)}{2}$. $\Gamma_2 \geq 0$ is satisfied.

# Examples on discrete groups

$G$ : discrete group
$\phi$: $G \to \mathbb{R}$, $\phi(e) = 0, \phi(g) = \phi(g^{-1})$ and conditionally negative .

$T_t(\lambda_g) = e^{\phi(g)t}\lambda_g$;
$\mathbb{R}[G]$: the algebra of all real valued bounded functions on $G$.

$$\langle \sum_g a_g \delta_g, \sum_h b_h \delta_h \rangle_\phi = \sum_{g,h} a_g a_h K_\phi(g,h)$$

with $K_\phi(g,h) = \frac{-\phi(gh^{-1}) + \phi(g) + \phi(h)}{2}$ defines a semi-inner product on $\mathbb{R}[G]$.
$N_\phi = \{x \in \mathbb{R}[G], \langle x, x \rangle_\phi = 0\}$,

$(T_t)_t$ satisfies all assumptions of Theorem 2 if

$$\dim \mathbb{R}[G]/N_\phi < \infty,$$

by Junge's reduction trick.