ON THE CONE MULTIPLIER IN $\mathbb{R}^3$

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The cone multiplier in $\mathbb{R}^3$

- The cone multiplier of order α

$$\widehat{\mathcal{C}^\alpha f}(\xi,\tau) = \left(1-\frac{|\xi|^2}{\tau^2}\right)_+^\alpha \phi(\tau)\,\widehat{f}(\xi,\tau), \quad (\xi,\tau)\in\mathbb{R}^2\times\mathbb{R},$$

where $\phi \in C_c^\infty[1/2,2]$.

- Cone multiplier conjecture (Stein, 1978): For $1 \le p \le \infty$

$$\|\mathcal{C}^\alpha f\|_{L^p(\mathbb{R}^3)} \le C\|f\|_{L^p(\mathbb{R}^3)},$$

if and only if

$$\alpha > \alpha(p) = \max\left(\left|1-\frac{2}{p}\right|-\frac{1}{2}, 0\right).$$

- By de Leeuw's restriction theorem this implies the sharp L^p boundedness of Bochner-Riesz operator.
- When $\alpha > \frac{1}{2}$, the kernel is in L^1. By Plancherel's theorem $\mathcal{C}^\alpha$ is bounded on L^2 for any $\alpha > 0$.
- Especially, when $\frac{4}{3} \le p \le 4$, $\alpha > 0$ is necessary.

• By finite decomposition and rotation (in the previous definition) we may assume that $\widehat{f}$ is supported in a small neighborhood of $(1, 0, 1)$. By a linear change of variables $(\xi_2, \xi_1 - \tau, \xi_1 + \tau) \to (\eta, \tau, \rho)$

$$\widehat{C^\alpha f}(\eta, \tau, \rho) = (\tau - \eta^2/\rho)_+^\alpha \phi(\eta, \tau, \rho)\widehat{f}(\eta, \tau, \rho), \quad \eta, \tau, \rho \in \mathbb{R},$$

and ϕ is a smooth function supported in a small neighborhood of $2e_3 = (0, 0, 2)$.

• Let $0 < \delta \ll 1$ and let $\psi \in C_c^\infty[1/2, 4]$ and define an operator C_δ by

$$\widehat{C_\delta f}(\eta, \tau, \rho) = \psi\left(\frac{\tau - \eta^2/\rho}{\delta}\right)\widehat{f}(\eta, \tau, \rho).$$

• **Sharp bound for C_δ:** For $\epsilon > 0$,

$$\|C_\delta f\|_p \le C\delta^{-\alpha(p)-\epsilon}\|f\|_p$$

for $\widehat{f}$ supported in the set $\{(\eta, \tau, \rho) : \rho \in [2^{-2}, 2^2],\ |\eta/\rho| \le 2^2\}$.

• $(\tau - \eta^2/\rho)_+^\alpha = \sum_{\delta:dyadic} \delta^\alpha \psi\left(\frac{\tau - \eta^2/\rho}{\delta}\right)$ by a proper choice of ψ.

Mockenhaupt's square function and Wolff's mixed norm estimates

Let $\phi \in C_c^\infty[-1,1]$ satisfying $\sum_{k\in\mathbb{Z}} \phi(k-\cdot) = 1$. For $\nu \in \sqrt{\delta}\mathbb{Z} \cap [-1,1]$, define a projection operator by

$$\widehat{S^\nu f} = \phi\Big(\frac{\nu - \eta/\rho}{\sqrt{\delta}}\Big)\widehat{f}.$$

• Sharp square function estimate:

$$\|\sum_\nu C_\delta S^\nu f\|_p \le C\delta^{-\alpha(p)/2-\epsilon}\|(\sum_\nu |C_\delta S^\nu f|^2)^{\frac{1}{2}}\|_p.$$

• $2 \le p \le \infty$. It was mostly studied with $p = 4$. We will show the sharp bound for $p = 3$. Square function can be controlled by Kakeya maximal bounds due to Cordoba:

$$\|(\sum_\nu |C_\delta S^\nu f|^2)^{\frac{1}{2}}\|_p \lesssim \delta^{-\alpha(p)/2-\epsilon}\|f\|_p.$$

• Mixed norm estimate: (only possible for $p \ge 6$)

$$\|\sum_\nu C_\delta S^\nu f\|_p \le C\delta^{-\alpha(p)-\epsilon}\|(\sum_\nu |C_\delta S^\nu f|^p)^{\frac{1}{p}}\|_p.$$

Progresses

- L^4 boundedness
  - (Mockenhaupt, 1993) $\alpha > \frac{1}{8}$

Square function and orthogonality in spirit of Fefferman's proof of Bochner-Riesz and Kakeya maximal estimates

  - (Bourgain, 1995) $\alpha > \frac{1}{8} - \epsilon$
  - (Tao-Vargas, 2000) $\alpha > \frac{1}{8} - \frac{1}{476}$

Systematic bilinear approach

  - (Wolff, 2001) $\alpha > \frac{1}{8} - \frac{1}{88}$

- Sharp L^p boundedness
  - (Wolff, 2000) $p > 74$

Mixed norm inequality associated with plate decomposition along with conical sector

  - (Garrigós-Seeger, 2009) $p > 63 + 1/3$
  - (Garrigós-Schlag-Seeger) $p > 20$ & L^4 bound, $\alpha > \frac{1}{9}$

Theorem (L-Vargas)

Suppose that $\widehat{f}$ is supported in $B(2e_3, 1)$. Then, for $\epsilon > 0$,

$$\|\sum_{\nu} C_\delta S^\nu f\|_3 \le C_\epsilon \delta^{-\epsilon} \|(\sum_{\nu} |C_\delta S^\nu f|^2)^{\frac{1}{2}}\|_3.$$

• Multilinear restriction estimate (Bennett-Carbery-Tao)

+ Adaptation of Bourgain-Guth argument

Theorem

For $\frac{3}{2} \le p \le 3$, $\|\mathcal{C}^\alpha f\|_p \le C\|f\|_p$ provided that $\alpha > 0$.

• Local smoothing conjecture for the wave equation: For $p \ge 2$, if $\beta > \alpha(p)$,

$$\|e^{it\sqrt{-\Delta}} f\|_{L^p_{x,t}([1,2]\times\mathbb{R}^2)} \le C\|f\|_{L^p_\beta}.$$

• Combining square function estimate and a stronger version of Kakeya maximal bound due to Mokenhaupt- Seeger-Sogge,

Corollary

Let $d = 2$ and $2 \le p \le 3$. Then the above holds for all $\beta > 0$.

Bourgain-Guth argument: mild scale analysis

• Let $\mathcal{A}(\lambda) = \{(\eta, \tau, \rho) : \rho \in [1/2, 7/2],\ |\eta/\rho| \le \lambda\}$.

• For $0 < \delta \ll 1$, define $\mathfrak{S}(\delta)$ to be the best constant for which

$$\|\sum_{\nu} C_\delta S^\nu f\|_3 \le \mathfrak{S}(\delta)\|(\sum_{\nu} |C_\delta S^\nu f|^2)^{\frac{1}{2}}\|_3$$

whenever $\operatorname{supp}\widehat{f} \subset \mathcal{A}(2)$. Trivially $\mathfrak{S}(\delta) \le C\delta^{-\frac{1}{4}}$. We aim to show

$$\mathfrak{S}(\delta) \le C\delta^{-\epsilon}, \quad \epsilon > 0.$$

• Bourgain-Guth argument for square function estimate

- Multilinear (trilinear) version of square function estimate
- Smaller Fourier support $\to$ improvement (by rescaling and stability)
- Decomposition: bound the operator with trilinear terms with transversality

while remaining parts has relatively small Fourier supports. *The decomposition here is much simpler because there are one parameter directional sets.*

Transversality of conic sectors

- Let Γ be a subset of the cone given by

$$\Gamma = \{(\eta, \tau, \rho) : \tau = \eta^2/\rho,\ \rho \in [3/2, 5/2], |\eta/\rho| \leq 3\}.$$

Define $\theta : \mathbb{R} \times \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ by $\theta(\eta, \tau, \rho) = \eta/\rho$. Identifying $\theta = \eta/\rho$ as an angular variable of (η, τ, ρ),

$$\Gamma = \{\rho(\theta, \theta^2, 1) :\ \rho \in [3/2, 5/2],\ |\theta| \leq 3\}.$$

- The normal vector to Γ at $(\eta, \tau, \rho) = \rho(\theta, \theta^2, 1)$ is parallel to

$$(2\eta/\rho, -1, -\eta^2/\rho^2) = (2\theta, -1, -\theta^2).$$

Given three points $(\eta_i, \tau_i, \rho_i) \in \Gamma, i = 1, 2, 3$, with angular variables $\theta_i = \eta_i/\rho_i$,

$$\det \begin{pmatrix} 2\theta_1 & -1 & -\theta_1^2 \\ 2\theta_2 & -1 & -\theta_2^2 \\ 2\theta_3 & -1 & -\theta_3^2 \end{pmatrix} = 2(\theta_1 - \theta_3)(\theta_1 - \theta_2)(\theta_2 - \theta_3).$$

• Three conical sectors are mutually transversal as long as they are are angularly separated. Hence it is possible to make use of the multilinear (trilinear) restriction estimate of Bennett-Cabery-Tao. Denote by $d\sigma$ the induced Lebesgue measure in Γ.

Proposition

Let Γ_1, Γ_2, and Γ_3 be subsets of Γ and $\epsilon_\circ > 0$. Suppose that $\theta(\Gamma_1)$, $\theta(\Gamma_2)$ and $\theta(\Gamma_3)$ are mutually separated by a distance $\gtrsim \epsilon_\circ > 0$. Let $R \gg \epsilon_\circ^{-1}$. Then, for $\epsilon > 0$, there is a constant $C_\epsilon = C_\epsilon(\epsilon_\circ)$ such that

$$\|\prod_{i=1}^{3} \widehat{g_i d\sigma}\|_{L^1(B_R)} \le C_\epsilon R^\epsilon \prod_{i=1}^{3} \|g_i\|_2$$

whenever g_i is supported in Γ_i, $i = 1, 2, 3$.

Suppose that $\widehat{F_i}$ is supported in $\Gamma_i + O(R^{-1})$, $i = 1, 2, 3$. Then, for $\epsilon > 0$ there is a constant $C_\epsilon = C_\epsilon(\epsilon_\circ)$ such that

$$\|\prod_{i=1}^{3} F_i\|_{L^1(B_R)} \le C_\epsilon R^{-\frac{3}{2}} R^\epsilon \prod_{i=1}^{3} \|F_i\|_2.$$

• Putting $\delta = R^{-1}$,

Lemma

Let $1 \gg \epsilon_\circ \gg \delta > 0$*. Suppose that* $\widehat{f_1}$, $\widehat{f_2}$, *and* $\widehat{f_3}$ *are supported in* $\mathcal{A}(3)$*. If* $\theta(\mathrm{supp}\,\widehat{f_1})$, $\theta(\mathrm{supp}\,\widehat{f_2})$, *and* $\theta(\mathrm{supp}\,\widehat{f_3})$ *are mutually separated by a distance* $\gtrsim \epsilon_\circ$, *then for* $\epsilon > 0$ *there is a constant* $C_\epsilon = C_\epsilon(\epsilon_\circ)$ *such that*

$$\|\prod_{i=1}^{3} C_\delta f_i\|_{L^1} \le C_\epsilon \delta^{\frac{3}{2}-\epsilon} \prod_{i=1}^{3} \|f_i\|_2.$$

Proposition

Under the same assumption as above, for $\epsilon > 0$ *there is a constant* $C_\epsilon = C_\epsilon(\epsilon_\circ)$ *such that*

$$\Big\| \prod_{i=1}^{3} (\sum_\nu C_\delta S^\nu f_i) \Big\|_{L^1} \le C_\epsilon \delta^{-\epsilon} \prod_{i=1}^{3} \Big\| (\sum_\nu |C_\delta S^\nu f_i|^2)^{\frac{1}{2}} \Big\|_{L^3}.$$

• Let ψ be a Schwartz function such that $\psi \geq 1$ on $B(0,1)$ with its Fourier transform supported in $B(0,1)$. Let $R = \delta^{-1}$ and set

$$\psi_z = \psi\Big(\frac{\cdot - z}{\sqrt{R}} \Big).$$

Let $z_0 \in \mathbb{R}^3$. $\mathcal{F}(\psi_{z_0}^2(\sum_\nu C_\delta S^\nu f_i))$ is supported in $\Gamma_i + O(R^{-1/2})$

$$\Big\| \prod_{i=1}^{3} \psi_{z_0}^2 (\sum_\nu C_\delta S^\nu f_i) \Big\|_{L^1} \leq C_\epsilon R^{-\frac{3}{4}+\epsilon} \prod_{i=1}^{3} \Big\| \psi_{z_0}^2 (\sum_\nu C_\delta S^\nu f_i) \Big\|_2.$$

• Note that the supports of $\mathcal{F}(\psi_{z_0}^2 C_\delta S^\nu f_i)$ are essentially disjoint. Hence by Hölder's inequality

$$\Big\| \prod_{i=1}^{3} \psi_{z_0}^2 (\sum_\nu C_\delta S^\nu f_i) \Big\|_{L^1} \leq C_\epsilon R^{-\frac{3}{4}+\epsilon} \prod_{i=1}^{3} \Big\| |\psi_{z_0}^2| (\sum_\nu |C_\delta S^\nu f_i|^2)^{\frac{1}{2}} \Big\|_{L^2}$$

$$\leq C_\epsilon R^{-\frac{3}{4}+\epsilon} R^{\frac{9}{2}(\frac{1}{2}-\frac{1}{3})} \prod_{i=1}^{3} \Big\| |\psi_{z_0}| (\sum_\nu |C_\delta S^\nu f_i|^2)^{\frac{1}{2}} \Big\|_{L^3}.$$

Rescaling and translation

By rescaling, smallness of support at Fourier side gives better bound. It is also true for the square function. Precisely,

Lemma

Let $0 < \delta \le \gamma^2 \le 1$*. Suppose that* $\widehat{f}$ *is supported in* $\mathcal{A}(2)$ *and the diameter of* $\theta(\mathrm{supp}\,\widehat{f}\,) \le \gamma$*, then*

$$\|\sum_{\nu} C_\delta S^\nu f\|_3 \le \mathfrak{S}(\delta/\gamma^2)\|(\sum_{\nu} |C_\delta S^\nu f|^2)^{\frac{1}{2}}\|_3.$$

For $\theta \in \mathbb{R}$ and $\nu \in \sqrt{\delta}\mathbb{Z}$, let us set

$$\mathcal{F}(S^{\nu,\theta} f) = \phi\Big(\frac{\nu + \theta - \eta/\rho}{\sqrt{\delta}}\Big)\widehat{f}(\eta, \tau, \rho).$$

• For any $\theta \in \mathbb{R}$ whenever $\widehat{f}$ is supported in $\mathcal{A}(1)$

$$\big\|\sum_{\nu} C_\delta S^{\nu,\theta} f\big\|_3 \le \mathfrak{S}(\delta)\big\|\big(\sum_{\nu} |C_\delta S^{\nu,\theta} f|^2\big)^{\frac{1}{2}}\big\|_3.$$

• Suppose $\widehat{f}$ is supported in $\mathcal{A}(2)$ and $\theta(\mathrm{supp}\,\widehat{f}\,) \subset [\theta_\circ - \gamma/2, \theta_\circ + \gamma/2]$. Set

$$T_{\theta_\circ,\gamma} = \begin{pmatrix} \gamma & 0 & \theta_\circ \\ 2\gamma\theta_\circ & \gamma^2 & \theta_\circ^2 \\ 0 & 0 & 1 \end{pmatrix}.$$

• By the change of variables $(\eta,\tau,\rho) \to T_{\theta_\circ,\gamma}(\eta,\tau,\rho)$

$$\widehat{C_\delta f}(T_{\theta_\circ,\gamma}(\eta,\tau,\rho)) = \psi\Big(\frac{(\tau-\eta^2)/\rho}{\delta\gamma^{-2}}\Big)\widehat{f}\,(T_{\theta_\circ,\gamma}(\eta,\tau,\rho)).$$

and $\mathrm{supp}\,(\widehat{f} \circ T_{\theta_\circ,\gamma}) \subset \mathcal{A}(1)$.

• Denoting by $T^*_{\theta_\circ,\gamma}$ the adjoint of $T_{\theta_\circ,\gamma}$,

$$C_\delta f(x) = C_{\delta/\gamma^2}(f \circ T^{*\ -1}_{\theta_\circ,\gamma})(T^*_{\theta_\circ,\gamma}x).$$

• Changing variables,

$$\|\sum_\nu C_\delta S^\nu f\|_3 = |\det T_{\theta_\circ,\gamma}|^{\frac{1}{3}}\|\sum_{k\in\sqrt{\delta/\gamma^2}\,\mathbb{Z}} C_{\delta/\gamma^2}[S^{k,\,-\gamma^{-1}\theta_\circ}](f \circ T^{*\ -1}_{\theta_\circ,\gamma})\|_3$$

$$\le \mathfrak{S}(\delta/\gamma^2)|\det T_{\theta_\circ,\gamma}|^{\frac{1}{3}}\|(\sum_{k\in\sqrt{\delta/\gamma^2}\mathbb{Z}} |C_{\delta/\gamma^2}[S^{k,\,-\gamma^{-1}\theta}](f \circ T^{*\ -1}_{\theta_\circ,\gamma})|^2)^{\frac{1}{2}}\|_3$$

$$= \mathfrak{S}(\delta/\gamma^2)\|(\sum_\nu |C_\delta S^\nu f|^2)^{\frac{1}{2}}\|_3.$$

Decomposition

- Let $\widehat{f}$ be supported in $\mathcal{A}(2)$. Recall $\sum_{\nu\in\sqrt{\delta}\mathbb{Z}} S^\nu f = f$.
- (Tow different scales) Let K_1, K_2 be dyadic numbers such that

$$1 \ll K_1 \ll K_2 \ll \delta^{-\epsilon},$$

$$\{\mathfrak{J}^1\} = \left\{\frac{k}{K_1} \in [-1,1] : \, k \in \mathbb{Z}\right\},$$
$$\{\mathfrak{J}^2\} = \left\{\frac{k}{K_2} \in [-1,1] : \, k \in \mathbb{Z}\right\}.$$

- Group $S^\nu f$ into functions $f_{\mathfrak{J}^2}$ by setting

$$f_{\mathfrak{J}^2} = \sum_{\nu\in(\mathfrak{J}^2-(2K_2)^{-1},\mathfrak{J}^2+(2K_2)^{-1}]} S^\nu f, \qquad f = \sum_{\mathfrak{J}^2} f_{\mathfrak{J}^2}.$$

- Group $f_{\mathfrak{J}^2}$ into functions $f_{\mathfrak{J}^1}$ by setting

$$f_{\mathfrak{J}^1} = \sum_{\mathfrak{J}^2\in(\mathfrak{J}^1-(2K_1)^{-1},\mathfrak{J}^1+(2K_1)^{-1}]} f_{\mathfrak{J}^2}, \qquad f = \sum_{\mathfrak{J}^1} f_{\mathfrak{J}^1}.$$

Preliminary decomposition

• Note

$$C_\delta f = \sum_{\mathfrak{J}_1^1} C_\delta f_{\mathfrak{J}_1^1}.$$

• Consider two cases

$$|C_\delta f(x)| \le 16 \max_{\mathfrak{J}^1} |C_\delta f_{\mathfrak{J}^1}(x)|, \qquad |C_\delta f(x)| \ge 16 \max_{\mathfrak{J}^1} |C_\delta f_{\mathfrak{J}^1}(x)|.$$

• First stage decomposition

$$|C_\delta f(x)| \le C \max_{\mathfrak{J}^1} |C_\delta f_{\mathfrak{J}^1}(x)| + CK_1 \max_{\mathfrak{J}_1^1, \mathfrak{J}_2^1 \colon |\mathfrak{J}_1^1 - \mathfrak{J}_2^1| \ge \frac{4}{K_1}} |C_\delta f_{\mathfrak{J}_1^1}(x) C_\delta f_{\mathfrak{J}_2^1}(x)|^{\frac{1}{2}}.$$

• One can apply bilinear estimate if there is one available. This will give linear estimate away from the endpoint case.

Control of $C_\delta f_{\mathfrak{J}_1^1}(x) C_\delta f_{\mathfrak{J}_2^1}(x)$

- Write

$$C_\delta f_{\mathfrak{J}_i^1} = \sum_{\mathfrak{J}_i^2 \in (\mathfrak{J}^1 - (2K_1)^{-1}, \mathfrak{J}^1 + (2K_1)^{-1}]} C_\delta f_{\mathfrak{J}_i^2}, \ i = 1, 2.$$

- Considering the cases

$$|C_\delta f_{\mathfrak{J}_1^1}(x) C_\delta f_{\mathfrak{J}_2^1}(x)| \le 2^5 \max_{\mathfrak{J}_1^2, \mathfrak{J}_2^2} |C_\delta f_{\mathfrak{J}_1^2}(x) C_\delta f_{\mathfrak{J}_2^2}(x)|,$$

$$|C_\delta f_{\mathfrak{J}_1^1}(x) C_\delta f_{\mathfrak{J}_2^1}(x)| \ge 2^5 \max_{\mathfrak{J}_1^2, \mathfrak{J}_2^2} |C_\delta f_{\mathfrak{J}_1^2}(x) C_\delta f_{\mathfrak{J}_2^2}(x)|,$$

- We may assume

$$|C_\delta f_{\mathfrak{J}_1^2}(x)| > K_2^{-100} \max_{\mathfrak{J}_2^2} |C_\delta f_{\mathfrak{J}_2^2}(x)|, \text{ and } |C_\delta f_{\mathfrak{J}_2^2}(x)| > K_2^{-100} \max_{\mathfrak{J}_1^2} |C_\delta f_{\mathfrak{J}_1^2}(x)|.$$

- Second stage decomposition

$$|C_\delta f_{\mathfrak{J}_1^1}(x) C_\delta f_{\mathfrak{J}_2^1}(x)| \le 2K_2^{-50} \max_{\mathfrak{J}^2 \in \{\mathfrak{J}_1^2\} \cup \{\mathfrak{J}_2^2\}} |C_\delta f_{\mathfrak{J}^2}(x)|^2 + 2^5 \max_{\mathfrak{J}_1^2, \mathfrak{J}_2^2} |C_\delta f_{\mathfrak{J}_1^2}(x) C_\delta f_{\mathfrak{J}_2^2}(x)|$$
$$+ K_2^{50} \max_{\substack{\mathfrak{J}_1^2, \mathfrak{J}_2^2, \mathfrak{J}_3^2 \in \{\mathfrak{J}_1^2\} \cup \{\mathfrak{J}_2^2\} \\ \min_{i \ne j} |\mathfrak{J}_i^2 - \mathfrak{J}_j^2| \ge \frac{2}{K_2}}} |C_\delta f_{\mathfrak{J}_1^2}(x) C_\delta f_{\mathfrak{J}_2^2}(x) C_\delta f_{\mathfrak{J}_3^2}(x)|^{\frac{2}{3}}.$$

Trilinear Control

- Combining the previous bounds,

$$|C_\delta f(x)| \leq C \max_{\mathfrak{J}^1} |C_\delta f_{\mathfrak{J}^1}(x)| + CK_1 \max_{\mathfrak{J}^2} |C_\delta f_{\mathfrak{J}^2}(x)|$$
$$+ K_2^{50} \max_{\substack{\mathfrak{J}_1^2, \mathfrak{J}_2^2, \mathfrak{J}_3^2 \\ \min_{i \neq j} |\mathfrak{J}_i^2 - \mathfrak{J}_j^2| \geq \frac{2}{K_2}}} |C_\delta f_{\mathfrak{J}_1^2}(x) C_\delta f_{\mathfrak{J}_2^2}(x) C_\delta f_{\mathfrak{J}_3^2}(x)|^{\frac{1}{3}}.$$

- Raising 3rd power and integrating,

$$\|C_\delta f\|_3^3 \leq C \sum_{\mathfrak{J}^1} \|C_\delta f_{\mathfrak{J}^1}\|_3^3 + CK_1^3 \sum_{\mathfrak{J}^2} \|C_\delta f_{\mathfrak{J}^2}\|_3^3$$
$$+ CK_2^{150} \sum_{\substack{\mathfrak{J}_1^2, \mathfrak{J}_2^2, \mathfrak{J}_3^2 \\ \min_{i \neq j} |\mathfrak{J}_i^2 - \mathfrak{J}_j^2| \geq \frac{2}{K_2}}} \|C_\delta f_{\mathfrak{J}_1^2} C_\delta f_{\mathfrak{J}_2^2} C_\delta f_{\mathfrak{J}_3^2}\|_1.$$

$$\sum_{\substack{\mathfrak{I}_1^2,\mathfrak{I}_2^2,\mathfrak{I}_3^2 \\ \min_{i\neq j}|\mathfrak{I}_i^2-\mathfrak{I}_j^2|\geq \frac{2}{K_2}}} \int |C_\delta f_{\mathfrak{I}_1^2}(x) C_\delta f_{\mathfrak{I}_2^2}(x) C_\delta f_{\mathfrak{I}_3^2}(x)|dx$$

$$\leq C_\epsilon\left(\frac{1}{K_2}\right)\delta^{-\epsilon}\prod_{i=1}^{3}\left\|(\sum_\nu |C_\delta f^\nu|^2)^{\frac{1}{2}}\right\|_3.$$

- Write $f_{\mathfrak{I}^1} = \sum_{\nu\sim\mathfrak{I}^1} S^\nu f$ so that

$$f = \sum_{\mathfrak{I}^1}\sum_{\nu\sim\mathfrak{I}^1} S^\nu f.$$

- Since the diameter of $\theta(f_{\mathfrak{I}_1}) \leq \frac{51}{50K_1}$ and set $c = 50^2/51^2$. Hence

$$\sum_{\mathfrak{I}^1}\|C_\delta f_{\mathfrak{I}^1}\|_3^3 = \sum_{\mathfrak{I}^1}\|\sum_{\nu\sim\mathfrak{I}^1} C_\delta S^\nu f\|_3^3 \leq [\mathfrak{S}(c\delta K_1^2)]^3 \sum_{\mathfrak{I}^1}\|(\sum_{\nu\sim\mathfrak{I}^1}|C_\delta S^\nu f|^2)^{\frac{1}{2}}\|_3^3$$

$$\leq [\mathfrak{S}(c\delta K_1^2)]^3\|(\sum_{\mathfrak{I}^1}\sum_{\nu\sim\mathfrak{I}^1}|C_\delta S^\nu f|^2)^{\frac{1}{2}}\|_3^3$$

$$\leq [\mathfrak{S}(c\delta K_1^2)]^3\|(\sum_{\nu}|C_\delta S^\nu f|^2)^{\frac{1}{2}}\|_3^3.$$

• Similarly,

$$\sum_{\mathfrak{J}^2} \|C_\delta f_{\mathfrak{J}^2}\|_3^3 \leq [\mathfrak{S}(c\delta K_2^2)]^3 \|(\sum_\nu |C_\delta S^\nu f|^2)^{\frac{1}{2}}\|_3^3.$$

Now combining all these estimates, we get

$$\|\sum_\nu C_\delta S^\nu f\|_3 \leq C\Big(\mathfrak{S}(c\delta K_1^2) + K_1 \mathfrak{S}(c\delta K_2^2) + K_2^{50} C_\epsilon \Big(\frac{1}{K_2}\Big)\delta^{-\epsilon}\Big) \times \|(\sum_\nu |C_\delta S^\nu f|^2)^{\frac{1}{2}}\|_3.$$

• Therefore

$$\mathfrak{S}(\delta) \leq C\Big(\mathfrak{S}(c\delta K_1^2) + K_1 \mathfrak{S}(c\delta K_2^2) + K_2^{50} C_\epsilon \Big(\frac{1}{K_2}\Big)\delta^{-\epsilon}\Big).$$

• Define

$$\overline{\mathfrak{S}}_\beta(\delta) = \sup_{\delta \leq \delta' \leq 1} (\delta')^\beta \mathfrak{S}(\delta').$$

Proof completes if $\overline{\mathfrak{S}}_\beta(\delta) \leq C$ for any $\beta > 0$.

- Let $\beta > 0$ and $\delta \le \delta_\circ \ll 1$. Then,

$$\delta_\circ^\beta \mathfrak{S}(\delta_\circ) \le C\Big(\delta_\circ^\beta \mathfrak{S}(c\delta_\circ K_1^2) + K_1 \delta_\circ^\beta \mathfrak{S}(c\delta_\circ K_2^2) + K_2^{50} C_\epsilon\Big(\frac{1}{K_2}\Big)\delta_\circ^\beta \delta_\circ^{-\epsilon}\Big)$$
$$\le C\Big(K_1^{-2\beta}(c\delta_\circ K_1^2)^\beta \mathfrak{S}(c\delta_\circ K_1^2) + K_1 K_2^{-2\beta}(c\delta_\circ K_2^2)^\beta \mathfrak{S}(c\delta_\circ K_2^2)$$
$$+ K_2^{50} C_\epsilon\Big(\frac{1}{K_2}\Big)\delta_\circ^{\beta-\epsilon}\Big)$$

Since $1 \le cK_2^2$, cK_1^2 and $\delta \le \delta_\circ$, taking $\epsilon = \beta$,

$$\delta_\circ^\beta \mathfrak{S}(\delta_\circ) \le C\Big(K_1^{-2\beta}\overline{\mathfrak{S}}_\beta(\delta) + K_1 K_2^{-2\beta}\overline{\mathfrak{S}}_\beta(\delta) + K_2^{50} C_\epsilon\Big(\frac{1}{K_2}\Big)\Big).$$

- Taking sup along $\delta_\circ \ge \delta$,

$$\overline{\mathfrak{S}}_\beta(\delta) \le C\Big(K_1^{-2\beta}\overline{\mathfrak{S}}_\beta(\delta) + K_1 K_2^{-2\beta}\overline{\mathfrak{S}}_\beta(\delta) + K_2^{50} C_\epsilon\Big(\frac{1}{K_2}\Big)\Big).$$

Finally, choose K_1, K_2 such that $CK_1^{-2\beta} < \frac{1}{4}$ and $CK_1K_2^{-2\beta} < \frac{1}{4}$ to get

$$\overline{\mathfrak{S}}_\beta(\delta) \le CK_2^{50} C_\epsilon\big(\frac{1}{K_2}\big).$$

Remarks

• As it is known, square function estimate can be used to improve sharp Wolff's inequality and improvement on Wolff's inequality also gives better square function. So, by the L^3 square function the range of L^4 estimate and sharp Wolff's inequality actually extends further.

• The argument here also works with the square function associated with spherical Bochner-Riesz operators. With additional work it is possible to give a different proof of result due Bourgain-Guth for Bochner-Riesz without relying on oscillatory integral estimate.

• For the higher dimensional cone multiplier the argument is more involved since we need to use lower dimensional restriction estimate when transversality fails. We don't know yet how to use multilinear restriction for the cone to get linear estimate. It is not clear even for restriction case.

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