Pointwise convergence of vector-valued Fourier series

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Work being reported

- T. H. & M. Lacey: "Pointwise convergence of vector-valued Fourier series" (arXiv, May 2012)

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quartile operator = a discrete model of Bilinear Hilbert Transform

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$d_0, d_1, \ldots, d_k, \ldots : \Omega \to X$ is a martingale difference sequence if

$$\int_\Omega \varphi(d_0, \ldots, d_{k-1}) d_k \, \mathrm{d}\mu = 0 \qquad \forall \varphi : X^k \to \mathbb{R}$$

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$$\Big\| \sum_{k=0}^{n} \varepsilon_k d_k \Big\|_{L^p(\Omega;X)} \leq C \Big\| \sum_{k=0}^{n} d_k \Big\|_{L^p(\Omega;X)}$$

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Theorem (Burkholder, Bourgain, Figiel; 1980's)

X is a UMD space, $1 < p < \infty$

- $\Leftrightarrow f \mapsto Hf(x) = \mathrm{p.v.} \int_{-\infty}^{\infty} \frac{f(y)}{x-y} \,\mathrm{d}y$ *is bounded on* $L^p(\mathbb{R}; X)$

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Research went on in the 1990's, 2000's, but one central classical theorem remained unextended...

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Other proofs: Fefferman (1973), Lacey & Thiele (2000).

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  - $S_N f(x) = O(\log \log N)$ a.e.

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Let $X = [Y, H]_\theta$, $Y \in UMD$, $H =$*Hilbert. Then*

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Partial sums:

$$S_N f(x) = \int_{-N}^{N} \hat{f}(\xi) e^{i2\pi\xi x} \, \mathrm{d}\xi = \Big(\int_{-\infty}^{N} - \int_{-\infty}^{-N} \Big) \hat{f}(\xi) e^{i2\pi\xi x} \, \mathrm{d}\xi.$$

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Not hard for $f \in \mathscr{S}(\mathbb{R})$ (dense in $L^p(\mathbb{R})$).

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To prove: $C : L^p(\mathbb{R}; X) \to L^{p,\infty}(\mathbb{R}; X)$.

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Wavepacket ϕ_P:

$$\operatorname{supp} \hat{\phi}_P \subseteq \omega_{P_d}, \qquad \operatorname{supp} \phi_P \underset{\approx}{\subset} I_P \qquad \text{(rapid decay outside).}$$

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"Iterated Calderón–Zygmund / atomic decomposition":

- Split $\mathbb{P}$ into a good part (controlled 'size') and a bad part (controlled 'support' + 'cancellation').
  $\mathbb{P} = \mathbb{B}_0 \cup \mathbb{G}_0$.

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- Eventually get a series of 'atoms' (controlled 'size' + 'support' + 'cancellation')

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Ok if $\#\{\text{trees } \mathbb{T}_j\} = 1$, since $f \mapsto \sum_{P\in\mathbb{T}} \langle f, \phi_P\rangle \phi_P$ is a
Calderón–Zygmund operator $\Rightarrow$ bounded on $L^q(\mathbb{R}; X)$ ($X \in \mathrm{UMD}$)

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Solution

Recall:

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The resulting bound is

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If $\bigstar$ holds $\forall\alpha < 1$, we say that X has **Fourier tile-type** q.

Fourier tile-type q of X

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- Looks like: Tile-type exponent q $<\approx>$ Admissible r for an r-variation Carleson theorem à la Oberlin–Seeger–Tao–Thiele–Wright (work in progress by I. Parissis in the Walsh model).

Thank you!

(Post scriptum in another file.)