Smoothing for the KdV equation and Zakharov system on the torus

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Joint work with Nikos Tzirakis (UIUC)

In this talk we consider the smoothing properties of certain dispersive PDE with periodic boundary conditions:

$$\begin{cases} u_t + L(u) + N(u) = 0, & x \in \mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z}), \quad t \in \mathbb{R}, \\ u(x,0) = g(x) \in H^s(\mathbb{T}). \end{cases}$$

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Question: Is the nonlinear Duhamel term smoother than the initial data (is it in H^{s+a} for some $a > 0$)?

The answer is affirmative for the Korteweg–de Vries (KdV) equation (with a smooth space-time potential V),

$$\begin{cases} u_t + u_{xxx} + uu_x + (Vu)_x = 0, \quad x \in \mathbb{T}, \quad t \in \mathbb{R}, \\ u(x,0) = g(x) \in H^s(\mathbb{T}), \end{cases}$$

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for modified KdV:

$$\begin{cases} u_t + u_{xxx} - u^2 u_x = 0, \quad x \in \mathbb{T}, \quad t \in \mathbb{R}, \\ u(x,0) = g(x) \in H^s(\mathbb{T}), \end{cases}$$

and for the Zakharov system which consists of a complex field u (Schrödinger part) and a real field n (wave part):

$$\begin{cases} iu_t + \alpha u_{xx} = nu, \quad x \in \mathbb{T}, \quad t \in \mathbb{R}, \\ \beta^{-2} n_{tt} - n_{xx} = (|u|^2)_{xx}, \\ u(x,0) = u_0(x) \in H^{s_0}(\mathbb{T}), \\ n(x,0) = n_0(x) \in H^{s_1}(\mathbb{T}), \quad n_t(x,0) = n_1(x) \in H^{s_1-1}(\mathbb{T}), \end{cases}$$

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B. E., N. Tzirakis, *Global smoothing for the periodic KdV evolution*, arXiv:1103.4190

B. E., N. Tzirakis, *Long time dynamics for forced and weakly damped KdV on the torus*, arXiv:1108.3358

B. E., N. Tzirakis, *Smoothing and global attractors for the Zakharov system on the torus*, arXiv:1202.5268

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$$u(x,0) = g(x), \qquad x \in \mathbb{R} \text{ or } \mathbb{T},\ t \in \mathbb{R},\ u(x,t) \in \mathbb{R},$$

describes surface water waves in the small amplitude limit of long waves in shallow water. It is completely integrable and has soliton solutions $\phi(x - vt)$, both on $\mathbb{R}$ and on $\mathbb{T}$.

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• Infinitely many conserved quantities:

$$\int u(x,0)dx = \int u(x,t)dx, \quad \text{momentum cons.},$$

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• On $\mathbb{T}$, one can work with mean-zero solutions by Galilean invariance:

$$u(x,t) \leftrightarrow u(x - ct, t) - c.$$

- Duhamel's formula:

$$u(x,t) = e^{-t\partial_x^3} g(x) - \int_0^t e^{-(t-t')\partial_x^3} [uu_x] dt',$$

$$e^{-t\partial_x^3} g(x) = \sum_{k\in\mathbb{Z}} e^{ikx} e^{ik^3 t} \widehat{g}(k), \qquad \widehat{g}(k) = \frac{1}{2\pi} \int_0^{2\pi} g(x) e^{-ikx} dx,$$

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• Problem: Integration in t should be converted to derivative gain in x

• On $\mathbb{R}$, high frequency components escape to $-\infty$ very fast. This causes smoothing:

$$\big\| \|\partial_x e^{-t\partial_x^3} g\|_{L^2_t} \big\|_{L^\infty_x} \leq C\|g\|_{L^2(\mathbb{R})} \quad \text{(Kato smoothing)},$$

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• Bourgain: $H^\epsilon(\mathbb{T}) \to L^6(\mathbb{T}^2)$.
Conjecture: $H^\epsilon(\mathbb{T}) \to L^8(\mathbb{T}^2)$.

- Bona, Smith (75): LWP in $H^s(\mathbb{T})$, $s > 3/2$. GWP for $s \geq 2$.
- Bourgain (93): GWP in $H^s(\mathbb{T})$, $s \geq 0$.
- Kenig, Ponce, Vega (96): LWP in $H^s(\mathbb{T})$, $s \geq -\frac{1}{2}$.
- Colliander, Keel, Staffilani, Takaoka, Tao (02): GWP in $H^s(\mathbb{T})$, $s \geq -\frac{1}{2}$.
- Kappeler, Topalov (04): GWP in $H^s(\mathbb{T})$, $s \geq -1$.

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We have

$$\frac{dx}{dt} = \frac{d}{dt}\Big(\frac{e^{i\Omega t} f(x)}{i\Omega}\Big) - \frac{e^{i\Omega t}}{i\Omega} f'(x) \frac{dx}{dt}$$

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If f and f' are bounded, we get

$$x(1) - x(0) = O(1/\Omega).$$

Bourgain's $X^{s,b}$ spaces

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Idea: space-time Fourier support of the nonlinear solution is concentrated around this set.

$$\|u\|_{X^{s,b}} = \big\|\langle k\rangle^s \langle \tau - k^3\rangle^b \widehat{u}(k,\tau)\big\|_{\ell^2_k L^2_\tau} = \|e^{t\partial_x^3} u\|_{H^s_x H^b_t}.$$

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For the local theory of KdV, one needs to work with $b = 1/2$.

$$\left\| \chi(t) \int_0^t e^{-(t-t')\partial_x^3}[uu_x]dt' \right\|_{X^{s,1/2}} \lesssim \|uu_x\|_{X^{s,-1/2}} \lesssim \|u\|^2_{X^{s,1/2}}.$$

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For $s = 0$, the multiplier is

$$\frac{|k_2|}{\langle \tau_1 - k_1^3 \rangle^{1/2} \langle \tau_2 - k_2^3 \rangle^{1/2} \langle \tau - k^3 \rangle^{1/2}}, \ \ \tau = \tau_1 + \tau_2, \ \ k = k_1 + k_2$$

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$$(\tau_1 - k_1^3) + (\tau_2 - k_2^3) - (\tau - k^3) = 3kk_1k_2.$$

Ignoring zero modes:

$$\max(\langle\tau_1 - k_1^3\rangle, \langle\tau_2 - k_2^3\rangle, \langle\tau - k^3\rangle) \geq |kk_1k_2| \gtrsim |k_2|^2.$$

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• Smoothing in the first Picard iteration:

$$e^{-t\partial_x^3} \int_0^t e^{t'\partial_x^3} \left[e^{-t'\partial_x^3} g \partial_x \left(e^{-t'\partial_x^3} g \right) \right] dt'$$

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On the Fourier side (ignoring zero modes):

$$\sum_{k_1+k_2=k}\int_0^t e^{-3ik_1k_2kt'} k_2 \widehat{g}(k_1)\widehat{g}(k_2) dt' = \sum_{k_1+k_2=k} \frac{\widehat{g}(k_1)\widehat{g}(k_2)}{-3ikk_1}(e^{-3ik_1k_2kt}-1).$$

Therefore, if $g \in L^2$, then the correction term is in H^1.

Theorem 1. (E., Tzirakis, 2011)

Let $s > -1/2$ *and* $a < \min(2s+1, 1)$. *Then, given* $g \in H^s$, *we have* $u - e^{-t\partial_x^3} g \in C_t^0 H_x^{s+a}$, *and*

$$\left\| u(t) - e^{-t\partial_x^3} g \right\|_{H^{s+a}} \leq C(\|g\|_{H^s}) \langle t \rangle^{\alpha(s)},$$

for some $\alpha(s) < \infty$.

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• Christ (04): $\mathcal{F}L^p \to \mathcal{F}L^q$ type smoothing for cubic NLS on $\mathbb{T}$.

Theorem 2 (E., Tzirakis, 2011).

Let $V \in C^\infty(\mathbb{T} \times \mathbb{R})$, and $\langle V \rangle = 0$ for each t. Consider the KdV equation with potential V:

$$u_t + u_{xxx} + (Vu)_x + uu_x = 0$$
$$u(x,0) = g(x), \qquad x \in \mathbb{T}, t \in \mathbb{R}$$

Let $s \geq 0$ and $a < 1$. Then, given $g \in H^s$, we have

$$u - e^{-t\partial_x^3} g \in C_t^0 H_x^{s+a}.$$

Moreover, we have some growth bounds for $\left\| u(t) - e^{-t\partial_x^3} g \right\|_{H^{s+a}}$.

• The growth rates in Theorem 1 and Theorem 2 depend on a priori growth bounds for the H^s norm. This implies growth bounds for H^{s+a} norm as follows.

$g \in H^{s+a} \implies$

$$\|u\|_{H^{s+a}} \leq \big\|u - e^{t\partial_x^3} g\big\|_{H^{s+a}} + \|e^{t\partial_x^3} g\|_{H^{s+a}} \leq C\big(\|g\|_{H^s}\big)\langle t\rangle^{\alpha(s)} + \|g\|_{H^{s+a}}.$$

• The growth rates in Theorem 1 and Theorem 2 depend on a priori growth bounds for the H^s norm. This implies growth bounds for H^{s+a} norm as follows.

$g \in H^{s+a} \implies$

$$\|u\|_{H^{s+a}} \leq \left\|u - e^{t\partial_x^3} g\right\|_{H^{s+a}} + \|e^{t\partial_x^3} g\|_{H^{s+a}} \leq C\big(\|g\|_{H^s}\big)\langle t\rangle^{\alpha(s)} + \|g\|_{H^{s+a}}.$$

• Staffilani (97): Polynomial growth bounds assuming that L^2 and H^1 norms remain bounded.

Theorem (Oskolkov).

If g is of bounded variation, then $e^{-t\partial_x^3}g$ is a bounded function for each t. Moreover,
$t/2\pi \notin \mathbb{Q} \implies e^{-t\partial_x^3}g$ *is continuous in* x,
if $t/2\pi \in \mathbb{Q} \implies e^{-t\partial_x^3}g$ *has at most countably many discontinuities.*
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• Theorems 1, 2 above imply that, for $g \in BV \subset L^2$,
$u - e^{-t\partial_x^3}g \in C_t^0 H_x^{1-} \subset C_t^0 C_x^0$.

Corollary. *Same for KdV (with a smooth space-time potential).*

Theorem (Hu & Li, 2011).

For $s > 3/14$,

$$\left\| e^{-t\partial_x^3} g \right\|_{L^{14}_{x,t\in\mathbb{T}}} \lesssim \|g\|_{H^s}.$$

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This (together with an argument by Moyua-Vega (08)) implies:

Corollary. *If* $g \in H^s$, $s > 3/7$, *then* $e^{-t\partial_x^3} g$ *converges to* g *almost everywhere as* $t \to 0$.

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Corollary. *Same for KdV (with a smooth space-time potential).*

• There are further applications to smoothing for modified KdV and to the existence of global attractors for forced and weakly damped KdV.

Write the equation on the Fourier side using

$$u(x,t) = \sum_{k \in \mathbb{Z}_0} u_k(t) e^{ikx}$$

with

$$u_k := \widehat{u}(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(t,x) e^{-ikx} dx$$

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Produces an infinite system of ordinary differential equations:

$$\partial_t u_k = -\frac{ik}{2} \sum_{k_1+k_2=k} u_{k_1} u_{k_2} + ik^3 u_k, \quad u_k(0) = \widehat{g}(k).$$

Since the solution is real valued, $\bar{u}_k = u_{-k}$.

Changing the variable $v_k(t) = u_k(t)e^{-ik^3 t}$, and using the identity

$$(k_1 + k_2)^3 - k_1^3 - k_2^3 = 3(k_1 + k_2)k_1 k_2,$$

we obtain

$$\partial_t v_k = -\frac{ik}{2} \sum_{k_1+k_2=k} e^{-i3kk_1k_2t} v_{k_1} v_{k_2}, \quad v_k(0) = \widehat{g}(k).$$

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Differentiation by parts and the equation yields

$$\partial_t \big(v_k + B_2(v)_k \big) = R_3(v)_k$$

where

$$B_2(v)_k = \frac{1}{6} \sum_{k_1+k_2=k} \frac{e^{-3ikk_1k_2t} v_{k_1} v_{k_2}}{k_1 k_2},$$

$$R_3(v)_k = -\frac{i}{6} \sum_{\substack{k_1+k_2+k_3=k \\ k_2+k_3\neq 0}} \frac{e^{-3it(k_1+k_2)(k_2+k_3)(k_3+k_1)}}{k_1} v_{k_1} v_{k_2} v_{k_3}.$$

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Resonant terms:

$$(k_1+k_2)(k_2+k_3)(k_3+k_1) = 0, \quad k_2+k_3\neq 0.$$

Write

$$R_3(v)_k = R_{3r}(v)_k + R_{3nr}(v)_k$$

$$\partial_t \left(v_k + B_2(v)_k \right) = R_{3r}(v)_k + R_{3nr}(v)_k.$$

$$R_{3r}(v)_k = \frac{i v_k |v_k|^2}{6k},$$

$$R_{3nr}(v)_k = -\frac{i}{6} \sum_{k_1+k_2+k_3=k}^{nr} \frac{e^{-3it(k_1+k_2)(k_2+k_3)(k_3+k_1)}}{k_1} v_{k_1} v_{k_2} v_{k_3}.$$

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• The range for a when $s \leq 0$ in Theorem 1 seems to be optimal up to the endpoint, since for general H^s data

$$|R_{3r}(v)_k| = (|v_k||k|^s)^3|k|^{-3s-1}$$

can not be in H^{s+a} if $a > 2s + 1$. This also implies that for $s = -1/2$ there is no smoothing within the tools that we use.

Now use the $X^{s,b}$ spaces of Bourgain:
Let

$$\mathcal{R}(u)(t,x) = \sum_{k \neq 0} R_{3nr}(u)_k(t) e^{ikx}.$$

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$$\mathcal{R}(u)(t,x) = \sum_{k\neq 0} R_{3nr}(u)_k(t)e^{ikx}.$$

Show

$$\|\mathcal{R}(u)\|_{X_\delta^{s+a,-1/2}} \lesssim \|u\|^3_{X_\delta^{s,1/2}}.$$

This requires Bourgain's periodic Strichartz: For any $\epsilon > 0$ and $b > 1/2$, we have

$$\|\chi_{[-\delta,\delta]}(t)u\|_{L^6_{t,x}(\mathbb{R}\times\mathbb{T})} \leq C_{\epsilon,b}\|u\|_{X_\delta^{\epsilon,b}}$$

Obtain the following energy inequality

$$\|u(t) - e^{-t\partial_x^3} g\|_{H^{s+a}} \lesssim \|u(t)\|_{H^s}^2 + \|g\|_{H^s}^2 + \int_0^t \|u(t')\|_{H^s}^3 dt' + \|u\|_{X_\delta^{s,1/2}}^3.$$

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Iterating this using the available H^s growth bounds yields the statement.

Forced and weakly damped KdV on the torus:

$$u_t + u_{xxx} + \gamma u + uu_x = f, \quad t \in \mathbb{R}, \; x \in \mathbb{T},$$

$$u(x,0) = g(x) \in \dot{L}^2(\mathbb{T}) := \left\{ h \in L^2(\mathbb{T}) : \int_{\mathbb{T}} h(x)dx = 0 \right\},$$

$\gamma > 0$ and $f \in \dot{L}^2$.

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$\gamma > 0$ and $f \in \dot{L}^2$.

$$\|u(t)\| \leq e^{-\gamma t}\|g\| + \frac{\|f\|}{\gamma}(1 - e^{-\gamma t}).$$

For $t > T = T(\gamma, \|g\|, \|f\|)$, we have $\|u(t)\| < 2\|f\|/\gamma$.

$B(0, 2\|f\|/\gamma) \subset L^2(\mathbb{T})$ is called an absorbing set.

Theorem 3. (E., Tzirakis, 2011)

Fix $s \in (0,1)$. *Consider the forced and weakly damped KdV equation on* $\mathbb{T} \times \mathbb{R}$ *with* $u(x,0) = g(x) \in \dot{L}^2$. *Then*

$$\left\| u(t) - e^{-\gamma t} e^{-t\partial_x^3} g \right\|_{H^s} \leq C(s, \gamma, \|g\|, \|f\|).$$

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$$\left\| u(t) - e^{-\gamma t} e^{-t\partial_x^3} g \right\|_{H^s} \leq C(s, \gamma, \|g\|, \|f\|).$$

Corollary

Fix $s \in (0,1)$. Consider the forced and weakly damped KdV equation on $\mathbb{T} \times \mathbb{R}$ with $u(x,0) = g(x) \in \dot{L}^2$. Then there exists $T = T(\gamma, \|g\|, \|f\|)$ such that for $t \geq T$,

$$\left\| u(t) - e^{-\gamma(t-T)} e^{-(t-T)\partial_x^3} u(T) \right\|_{H^s} \leq C(s, \gamma, \|f\|).$$

• For any $s \in (0,1)$, all $\dot{L}^2$ solutions are attracted by a ball in H^s centered at zero of radius depending only on $s, \gamma, \|f\|$.

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• The description of the dynamics is explicit. After time T the evolution can be written as a sum of the linear evolution which decays to zero exponentially and a nonlinear evolution contained by the attracting ball.

Definition

A Global Attractor for a semigroup $\{U(t)\}_{t\geq 0}$ on a Hilbert space H is a compact set $\mathcal{A} \subset H$ which is invariant under the flow and which attracts all solutions:

$$\text{For all } g \in H, d(U(t)g, \mathcal{A}) \to 0, \text{ as } t \to \infty.$$

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Existence and regularity of the global attractor for forced damped KdV:

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Theorem 4. (E., Tzirakis, 2011)

Consider the forced and weakly damped KdV equation on $\mathbb{T} \times \mathbb{R}$. Then the equation possesses a global attractor in $\dot{L}^2$. Moreover, for any $s \in (0, 1)$, the global attractor is a compact subset of H^s bounded by a constant that depends only on s, γ, and $\|f\|$.

• New information: Explicit bound of the attractor set in H^s depending on s, γ, and $\|f\|$.

• Proof is simpler than the previous known proofs on the existence of the attractor.

• All higher order Sobolev norms for the forced and weakly damped KdV remain bounded for positive times.

• In the case $\gamma = 0$, all Sobolev norms grow at most polynomially.

$$\begin{cases} iu_t + \alpha u_{xx} = nu, \quad x \in \mathbb{T}, \quad t \in \mathbb{R}, \\ n_{tt} - n_{xx} = (|u|^2)_{xx}, \\ u(x,0) = u_0(x) \in H^{s_0}(\mathbb{T}), \\ n(x,0) = n_0(x) \in H^{s_1}(\mathbb{T}), \quad n_t(x,0) = n_1(x) \in H^{s_1-1}(\mathbb{T}), \end{cases}$$

Zakharov system describes the propagation of Langmuir waves in an ionized plasma.
Langmuir waves: rapid oscillations of the electron density in a conducting media.

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Energy space: $s_0 = 1$, $s_1 = 0$.

- Bourgain (94): LWP in the energy space, $\alpha = 1$.

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• Takaoka (99): LWP for $s_1 \geq 0$ and $\max(s_1, \frac{s_1}{2} + \frac{1}{2}) \leq s_0 \leq s_1 + 1$ when $\frac{1}{\alpha} \in \mathbb{N}$,
LWP for $s_1 \geq -\frac{1}{2}$, $\max(s_1, \frac{s_1}{2} + \frac{1}{4}) \leq s_0 \leq s_1 + 1$ when $\frac{1}{\alpha} \notin \mathbb{N}$.

Dissipative Zakharov system in the energy space.

$$\begin{cases} iu_t + \alpha u_{xx} + i\gamma u = nu + f, \quad x \in \mathbb{T}, \quad t \in [-T, T], \\ n_{tt} - n_{xx} + \delta n_t = (|u|^2)_{xx} + g, \\ u(x,0) = u_0(x), \ \ n(x,0) = n_0(x), \ \ n_t(x,0) = n_1(x). \end{cases} \tag{1}$$

where $f \in H^1(\mathbb{T}),\ g \in L^2(\mathbb{T})$ are time-independent, $\int_{\mathbb{T}} g(x)dx = 0$, and the damping coefficients $\delta,\ \gamma > 0$.

Theorem 5. (E., Tzirakis, 2012)

Suppose $\frac{1}{\alpha} \notin \mathbb{N}$. Consider the solution of the Zakharov system with $(u_0, n_0, n_1) \in H^{s_0} \times H^{s_1} \times H^{s_1-1}$. Assume that we have a growth bound

$$\|u\|_{H^{s_0}} + \|n\|_{H^{s_1}} + \|n_t\|_{H^{s_1-1}} \lesssim (1+|t|)^{\gamma(s_0,s_1)}. \tag{2}$$

Then, for any $a_0 \leq \min(1, 2s_0, 1+2s_1)$ (the inequality has to be strict if $s_0 - s_1 = 1$) and for any $a_1 \leq \min(1, 2s_0, 2s_0 - s_1)$, we have

$$u(t) - e^{i\alpha t \partial_x^2} u_0 \in C_t^0 H_x^{s_0+a_0}, \tag{3}$$

$$(n, n_t) - \Phi_t(n_0, n_1) \in C_t^0 (H_x^{s_1+a_1} \times H_x^{s_1-1+a_1}), \tag{4}$$

where Φ_t is the propagator of linear wave equation. Moreover, for $\beta > 1 + 15\gamma(s_0, s_1)$, we have

$$\|u(t) - e^{i\alpha t \partial_x^2} u_0\|_{H^{s_0+a_0}} + \|(n, n_t) - \Phi_t(n_0, n_1)\|_{H^{s_1+a_1} \times H^{s_1-1+a_1}} \lesssim (1+|t|)^{\beta}. \tag{5}$$

Theorem 6. (E., Tzirakis, 2012)

Suppose $\frac{1}{\alpha} \in \mathbb{N}$, and (2) hold. Then, for any $a_0 \leq \min(1, s_1)$ (the inequality has to be strict if $s_0 - s_1 = 1$ and $s_1 \geq 1$) and for any $a_1 \leq \min(1, 2s_0 - s_1 - 1)$, we have (3), (4) and (5).

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Corollary

For any $s_0 \geq 1$, $s_1 \geq 0$, the global solution of the Zakharov system with $H^{s_0} \times H^{s_1} \times H^{s_1 - 1}$ data satisfies the growth bound

$$\|u\|_{H^{s_0}} + \|n\|_{H^{s_1}} + \|n_t\|_{H^{s_1-1}} \leq C_1(1 + |t|)^{C_2},$$

where C_1 depends on s_0, s_1, and $\|u_0\|_{H^{s_0}} + \|n_0\|_{H^{s_1}}, \|n_1\|_{H^{s_1-1}}$, and C_2 depends on s_0, s_1.

Without loss of generality in the dissipative Zakharov we set $\gamma = \delta$ and $g = 0$.

$$\begin{cases} iu_t + \alpha u_{xx} + i\gamma u = nu + f, \quad x \in \mathbb{T}, \quad t \in (0, \infty), \\ n_{tt} - n_{xx} + \gamma n_t = (|u|^2)_{xx}, \\ u(x,0) = u_0(x), \;\; n(x,0) = n_0(x), \;\; n_t(x,0) = n_1(x). \end{cases} \tag{6}$$

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Theorem 7. (E., Tzirakis, 2012)

Consider the dissipative Zakharov system on $\mathbb{T} \times [0,\infty)$ with $u_0 \in H^1$ and with mean-zero $n_0 \in L^2$, $n_1 \in H^{-1}$. Then the equation possesses a global attractor in $H^1 \times \dot{L}^2 \times \dot{H}^{-1}$. Moreover, for any $a \in (0,1)$, the global attractor is a compact subset of $H^{1+a} \times H^a \times H^{-1+a}$, and it is bounded in $H^{1+a} \times H^a \times H^{-1+a}$ by a constant depending only on a, α, γ, and $\|f\|_{H^1}$.