Finite time singularities for incompressible fluids

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In collaboration with Ángel Castro, Charles Fefferman, Francisco Gancedo, María López-Fernández and Javier Gómez-Serrano

Equation

$$\begin{array}{l}\rho_t + u \cdot \nabla\rho = 0,\\ \nabla \cdot u = 0,\end{array} \qquad \rho(x_1, x_2, t) = \begin{cases} \rho^1, & x \in \Omega^1(t) \\ \rho^2, & x \in \Omega^2(t) = R^2 \setminus \Omega^1(t) \end{cases}$$

1. Muskat:

$$\frac{\mu}{\kappa} u = -\nabla p - (0, \mathrm{g}\rho) \quad \text{Darcy's law (Hele Shaw cell).}$$

2. Water wave:

$$\rho(u_t + (u \cdot \nabla)u) = -\nabla p - (0, \mathrm{g}\rho), \quad \text{Euler.}$$

with $\nabla \times u = 0$, $\rho^1 = 0$ and $\rho^2 \neq 0$.

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Scenario

We consider:

1. open curves vanishing at infinity

$$\lim_{\alpha\to\infty}(z(\alpha,t)-(\alpha,0))=0,$$

2. periodic curves in the space variable

$$z(\alpha+2k\pi,t)=z(\alpha,t)+2k\pi(1,0).$$

3. closed contours

$$z(\alpha+2k\pi,t)=z(\alpha,t).$$

Equations

Muskat

$$\begin{aligned} z_t(\alpha,t) &= BR(z,\varpi)(\alpha,t) + c(\alpha,t)\partial_\alpha z(\alpha,t), \\ \varpi(\alpha,t) &= -\mathrm{g}\kappa(\rho^2-\rho^1)\partial_\alpha z_2(\alpha,t). \end{aligned}$$

Water Wave

$$\begin{aligned} z_t(\alpha,t) &= BR(z,\varpi)(\alpha,t) + c(\alpha,t)\partial_\alpha z(\alpha,t), \\ \varpi_t(\alpha,t) &= -2\partial_t BR(z,\varpi)\cdot\partial_\alpha z - \partial_\alpha\Big(\frac{|\varpi|^2}{4|\partial_\alpha z|^2}\Big) + \partial_\alpha(c\,\varpi) \\ &\quad + 2c\,\partial_\alpha BR(z,\varpi)\cdot\partial_\alpha z(\alpha,t) - 2\mathrm{g}\partial_\alpha z_2, \end{aligned}$$

where $BR(z,\varpi) = \frac{1}{2\pi} PV \int_{\mathbb{R}} \frac{(z(\alpha,t)-z(\beta,t))^\perp}{|z(\alpha,t)-z(\beta,t)|^2}\varpi(\alpha,t)d\alpha$

Rayleigh-Taylor condition

A linearization around a flat contour $(\alpha, \epsilon f(\alpha, t))$, allows us to find

$$f_t = \frac{1}{2} H(\omega)$$

The equations

$$\omega = \sigma \partial_\alpha f, \qquad \text{(linear Muskat)}$$
$$\omega_t = \sigma \partial_\alpha f, \qquad \text{(linear water waves)}$$

show the parabolicity of the Muskat problem and the dispersive character of water waves.

- Rayleigh-Taylor condition:

$$\sigma(\alpha, t) = -(\nabla p^2(z(\alpha, t), t) - \nabla p^1(z(\alpha, t), t)) \cdot \partial_\alpha^\perp z(\alpha, t) > 0,$$

Local existence

- Local existence for initial data satisfying $z_0(\alpha) \in H^k$ and $\varpi_0(\alpha) \in H^{k-1}$,

$$\mathcal{F}(z_0)(\alpha,\beta) < \infty, \quad \text{and} \quad \sigma(\alpha,0) > 0.$$

1. Muskat: $k \geq 3$ joint work with F. Gancedo (2007),.....
2. Water Wave: $k \geq 4$ S. Wu (1997), Lannes, Christodoulou-Lindblad, Lindblad, Ambrose-Masmoudi, Coutand-Shkoller, Shatah- Zeng, Zhang-Zhang, Cordoba-Cordoba-Gancedo, Alazard-Burq-Zuily,....

where

$$\mathcal{F}(z)(\alpha,\beta,t) = \frac{|\beta|}{|z(\alpha,t) - z(\alpha-\beta,t)|} \qquad \forall\, \alpha,\beta \in (-\pi,\pi),$$

and

$$\mathcal{F}(z)(\alpha,0,t) = \frac{1}{|\partial_\alpha z(\alpha,t)|}.$$

Muskat

Then, by choosing an appropriate term c the dynamics of the interface satisfies

Equation

$$z_t(\alpha,t)=\frac{\rho^2-\rho^1}{2\pi}PV\int\frac{(z_1(\alpha,t)-z_1(\beta,t))}{|z(\alpha,t)-z(\beta,t)|^2}(\partial_\alpha z(\alpha,t)-\partial_\alpha z(\beta,t))d\beta.$$

Rayleigh-Taylor:

$$\sigma(\alpha,t)=\mathrm{g}(\rho^2-\rho^1)\partial_\alpha z_1(\alpha,t).$$

A wise parametrization: $\partial_\alpha z_1(\alpha,t)=1.$

We have $z(\alpha,t)=(\alpha,f(\alpha,t))$ which implies

$$F(z)(\alpha,\beta)\leq 1.$$

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Contour equation

- $\rho^2 > \rho^1$ denser fluid below (stable)
- $\rho^2 < \rho^1$ denser fluid above (unstable)

$$f_t(\alpha,t) = \mathrm{g}\frac{\rho^2-\rho^1}{2\pi} PV \int_{\mathbb{R}} \frac{(\partial_\alpha f(\alpha,t) - \partial_\alpha f(\alpha-\beta,t))\beta}{\beta^2 + (f(\alpha,t) - f(\alpha-\beta,t))^2} d\beta,$$
$$f(\alpha,0) = f_0(\alpha), \quad \alpha \in \mathbb{R}.$$

- L^2 maximum principle:

$$\|f\|_{L^2}^2(T) + \frac{\rho^2-\rho^1}{2\pi}\int_0^T\int_{\mathbb{R}}\int_{\mathbb{R}} \ln\Big(1+\Big(\frac{f(x,t)-f(\alpha,t)}{x-\alpha}\Big)^2\Big) dx d\alpha dt = \|f_0\|_{L^2}^2.$$

Bound

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \ln\Big(1+\Big(\frac{f(x,t)-f(\alpha,t)}{x-\alpha}\Big)^2\Big) dx d\alpha \le 4\pi\sqrt{2}\|f\|_{L^1}(t).$$

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$$=\|f_0\|^2_{L^2}.$$

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Global existence

- Muskat
  - Maximum Principle: $\|f\|_{L^\infty}(t) \leq \|f_0\|_{L^\infty}$
  - Global existence: $\sum |\xi||\hat{f}_0(\xi)| \leq 1/5$
  - Global existence: $\|\partial_x f_0\|_{L^\infty} < 1$

in collaboration with P. Constantin, F. Gancedo & R. Strain (2011).

- Water waves
  - Exponential in time existence in 2D for small initial data. S. Wu (2009)
  - Global existence in 3D for small initial data

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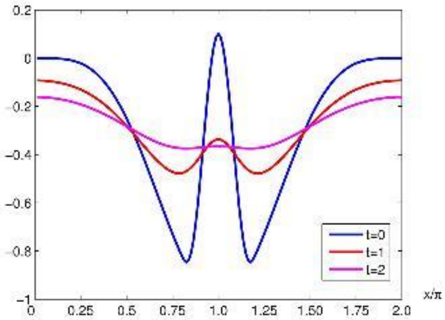
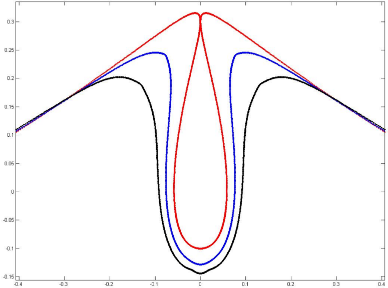
Singularities?: Numerical simulations of Muskat



Numerical simulations of María López-Fernández:

Singularities?: Numerical simulations of Water waves



Numerical simulations of Javier Gómez-Serrano:

Singularities for Muskat: Theorems

Joint work with A.Castro, C. Fefferman, F. Gancedo and M. López-Fernandez.

Theorem: Turning waves. R-T breakdown

There exists a non-empty open set of initial data H^4, satisfying the R-T (strictly positive $\sigma > 0$) for which the R-T of the solution of the Muskat problem breaks down in finite time.

Theorem: Breakdown of smoothness

There exists a non-empty open set of analytic initial data in the stable regime such that the solution turns to the unstable regime and later it breaks down.

- Applications to water waves.

Singularities for Water waves: Theorems

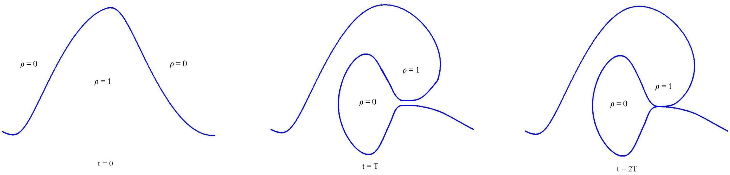
Joint work with A.Castro, C. Fefferman, F. Gancedo and J. Gómez-Serrano.

Theorem: Splash singularity

There exists a non-empty open set of smooth initial data for which the solution of water waves develops a splash singularity in finite.

Theorem: Stability from the splash

Given an approximate solution $(x(\alpha, t), \gamma(\alpha, t))$ *of water waves (up to the splash) then near* $(x(\alpha, t), \gamma(\alpha, t))$ *there exists an exact solution* $(z(\alpha, t), \omega(\alpha, t))$ *of water waves.*



Steps of the proof of R-T breakdown

- Instant analyticity if $\sigma > 0$.

$$S(t) = \{\alpha + i\zeta \in \mathbb{C} : \ \alpha \in \mathbb{T}, \ |\zeta| < ct\},$$

$$\|z\|^2_{H^k(S)}(t) = \|z\|^2_{L^2(S)}(t) + \sum_{\pm} \int_{\mathbb{T}} |\partial^k_\alpha z(\alpha \pm ict, t)|^2 d\alpha,$$

$$\frac{d}{dt}\|z\|_{H^k(S)}(t) \leq C(\|z\|_{H^k(S)}(t) + 1)^k.$$

Steps of the proof of R-T breakdown

- The region of analyticity does not collapse to the real axis as long as the $\sigma \geq 0$.

$$S(t) = \{\alpha + i\zeta \in \mathbb{C} \, : \, |\zeta| < h(t)\}, \qquad h(0) > 0$$

$$\frac{d}{dt} \sum_{\pm} \int_{\mathbb{T}} |\partial_\alpha^4 z_\mu(\alpha \pm ih(t))|^2 d\alpha \leq C(\|z\|_S(t) + 1)^k$$

$$+ (C(\|z\|_S(t)+1)^k h(t) + \frac{1}{10} h'(t)) \int_{\mathbb{T}} \Lambda(\partial_\alpha^4 z_\mu)(\alpha \pm ih(t)) \cdot \overline{\partial_\alpha^4 z_\mu(\alpha \pm ih(t))} d\alpha.$$

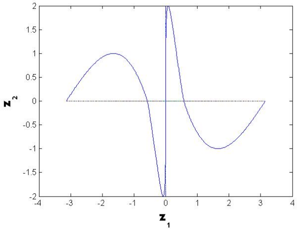
Choosing

$$h(t) = h(0) \exp(-10C \int_0^t (\|z\|_S + 1)^k (r) dr)$$

And we obtain finally

$$\frac{d}{dt} \sum_{\pm} \int_{\mathbb{T}} |\partial_\alpha^4 z(\alpha \pm ih(t))|^2 d\alpha \leq C(\|z\|_S(t) + 1)^{k+2}.$$

Steps of the proof of R-T breakdown



- There exists a curve $z(\alpha) = (z_1(\alpha), z_2(\alpha))$ with the following properties:

1. $z_1(\alpha) - \alpha$ and $z_2(\alpha)$ are analytic $2\pi -$ *periodic* functions and $z(\alpha)$ satisfies the arc-chord condition,
2. $z(\alpha)$ is odd and
3. $\partial_\alpha z_1(\alpha) > 0$ if $\alpha \neq 0$, $\partial_\alpha z_1(0) = 0$ and $\partial_\alpha z_2(0) > 0$,

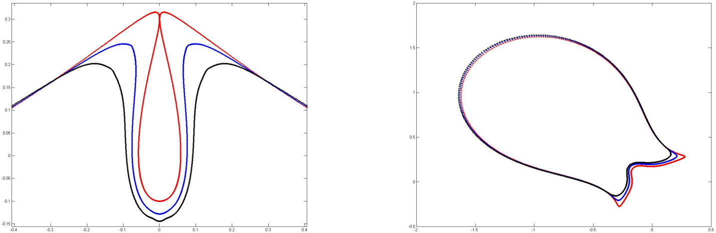
such that

$$(\partial_\alpha v_1)(0) < 0.$$

Steps of the proof of R-T breakdown

- Together with Cauchy-Kovalevsky theorem and a perturbative argument allows us to conclude that the unstable regime is reached for a curve initially in H^4.
- We construct a curve in the unstable regime which is analytic except in a single point. We show that as we evolve backwards in time the curve becomes analytic and is as close as we desired to the curve that we previously showed that it turns.

Steps of the proof of splash singularity



$$P(w) = \left(\tan\left(\frac{w}{2}\right)\right)^{1/2}, \quad w \in \mathbb{C},$$

- The water wave equations are invariant under time reversal. To obtain a solution that ends in a splash, we can therefore take our initial condition to be a splash, and show that there is a smooth solution for small times $t > 0$.

Steps of the proof of splash singularity

- Equations in the new domain

$$\tilde{z}_t(\alpha,t) = Q^2(\alpha,t)BR(\tilde{z},\tilde{\omega})(\alpha,t) + \tilde{c}(\alpha,t)\tilde{z}_\alpha(\alpha,t)$$

$$\tilde{\omega}_t(\alpha,t) = -2\partial_t BR(\tilde{z},\tilde{\omega})(\alpha,t)\cdot\tilde{z}_\alpha(\alpha,t) - |BR(\tilde{z},\tilde{\omega})|^2\partial_\alpha Q^2(\alpha,t)$$
$$-\partial_\alpha\left(\frac{Q^2(\alpha,t)}{4}\frac{\tilde{\omega}(\alpha,t)^2}{|z_\alpha(\alpha,t)|^2}\right) + 2\tilde{c}(\alpha,t)\partial_\alpha BR(\tilde{z},\tilde{\omega})\cdot\tilde{z}_\alpha(\alpha,t)$$
$$+\partial_\alpha\left(\tilde{c}(\alpha,t)\tilde{\omega}(\alpha,t)\right) - 2\partial_\alpha\left(P_2^{-1}(\tilde{z}(\alpha,t))\right).$$

where

$$Q^2(\alpha,t) = \left|\frac{dP}{dw}(z(\alpha,t))\right|^2.$$

Steps of the proof of splash singularity

Theorem: Local existence

Let $z^0(\alpha)$ be a splash curve such that $z_1^0(\alpha) - \alpha, z_2^0(\alpha) \in H^4(\mathbb{T})$. Let $u^0(\alpha) \cdot (z_\alpha^0)^\perp(\alpha) \in H^4(\mathbb{T})$ satisfying:

1. $u^0(\alpha_1) \cdot \dfrac{(z_\alpha^0)^\perp}{|z_\alpha^0|} < 0, u^0(\alpha_2) \cdot \dfrac{(z_\alpha^0)^\perp}{|z_\alpha^0|} < 0.$
2. $\displaystyle\int_{\partial\Omega} u^0 \cdot \frac{(z_\alpha^0)^\perp}{|z_\alpha^0|} ds = \int_{\mathbb{T}} u^0(\alpha) \cdot (z_\alpha^0)^\perp d\alpha = 0.$

Then there exist a finite time $T > 0$, a curve $\tilde{z}(\alpha, t) = P(z(\alpha, t)) \in C([0, T]; H^4)$ satisfying:

1. $P^{-1}(\tilde{z}_1(\alpha, t)) - \alpha, P^{-1}(\tilde{z}_2(\alpha, t))$ *are* 2π*-periodic,*
2. $P^{-1}(\tilde{z}(\alpha, t))$ *satisfies the arc-chord condition for all* $t \in (0, T]$*,*

and $\tilde{u}(\alpha, t) \in C([0, T]; H^3(\mathbb{T}))$ which provides a solution of the water waves equations in the new domain $\tilde{z}^0(\alpha) = P(z^0(\alpha))$.

Steps of the proof of splash singularity

$$E(t) = \|\tilde{z}\|^2_{H^3}(t) + \int_{\mathbb{T}} \frac{Q^2\sigma_{\tilde{z}}}{|\tilde{z}_\alpha|^2}|\partial^4_\alpha \tilde{z}|^2 d\alpha(t) + \|F(\tilde{z})\|^2_{L^\infty}(t)$$

$$+ \|\tilde{\omega}\|^2_{H^2}(t) + \|\varphi\|^2_{H^{3+\frac{1}{2}}}(t) + \frac{|\tilde{z}_\alpha|^2}{m(Q^2\sigma_{\tilde{z}})(t)} + \sum_{l=0}^{4} \frac{1}{m(q_l)(t)}$$

where

$$\sigma_{\tilde{z}} \equiv \left(BR_t(\tilde{z},\tilde{\omega}) + \frac{\varphi}{|\tilde{z}_\alpha|}BR_\alpha(\tilde{z},\tilde{\omega})\right)\cdot \tilde{z}_\alpha^\perp + \frac{\tilde{\omega}}{2|\tilde{z}_\alpha|^2}\left(\tilde{z}_{\alpha t} + \frac{\varphi}{|\tilde{z}_\alpha|}\tilde{z}_{\alpha\alpha}\right)\cdot \tilde{z}_\alpha^\perp$$

$$+ Q\left|BR(\tilde{z},\tilde{\omega}) + \frac{\tilde{\omega}}{2|\tilde{z}_\alpha|^2}\tilde{z}_\alpha\right|^2 (\nabla Q)(\tilde{z})\cdot \tilde{z}_\alpha^\perp - (\nabla P_2^{-1})(\tilde{z})\cdot \tilde{z}_\alpha^\perp$$

$$\varphi(\alpha,t) = \frac{Q^2(\alpha,t)\tilde{\omega}(\alpha,t)}{2|\tilde{z}_\alpha(\alpha,t)|} - \tilde{c}(\alpha,t)|\tilde{z}_\alpha(\alpha,t)|.$$

$$c(\alpha,t) = \frac{\alpha+\pi}{2\pi}\int_{-\pi}^{\pi}(Q^2 BR(\tilde{z},\tilde{\omega}))_\beta(\beta,t)\cdot \frac{\tilde{z}_\beta(\beta,t)}{|\tilde{z}_\beta(\beta,t)|^2}d\beta$$

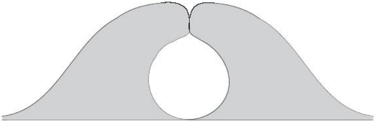
$$- \int_{-\pi}^{\alpha}(Q^2 BR(\tilde{z},\tilde{\omega}))_\beta(\beta,t)\cdot \frac{\tilde{z}_\beta(\beta,t)}{|\tilde{z}_\beta(\beta,t)|^2}d\beta$$

Further Results

- Splat

A Variant of the Splash:

SPLAT!

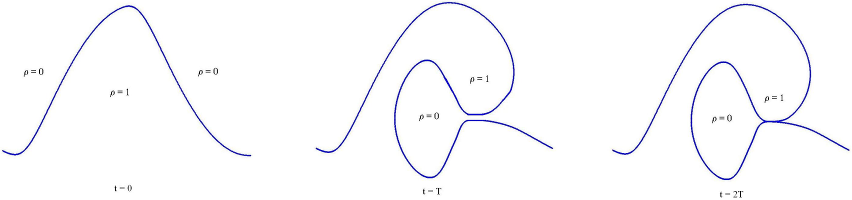


At time t_2, the interface self-intersects along an arc, but u and $\partial\Omega$ are otherwise smooth.

- Surface tension

Further Research

We would like to prove (in the near future) that starting from a graph, we get to a splash



Graph to Splash: Sketch of the proof

- Compute the constant in the stability theorem, i.e. quantify how fast solutions with near starting conditions separate.
- From a given solution obtained by simulation, calculate (using a computer!!) rigorous bounds in some H^k norm on how well the candidate satisfies the equation.
- By the stability theorem, there should be a function which solves the water waves equation, is a graph at time 0 and a splash at time T which is close enough to the candidate.

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