Existence of frames with prescribed norms and frame operator

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Statement of problem

Definition

A sequence $\{f_i\}_{i\in I}$ *in a Hilbert space* $\mathcal{H}$ *is called a frame if there exist constants* $0 < A \le B < \infty$ *such that*

$$A\|f\|^2 \le \sum_{i\in I} |\langle f, f_i\rangle|^2 \le B\|f\|^2 \qquad \forall f \in \mathcal{H}.$$

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Trivial necessary condition:

$$0 \le ||f_i||^2 \le B.$$

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Orthonormal dilation of Parseval frames

Theorem (Han, Larson (2000))

Let $\mathcal{K}$ be a Hilbert space with orthonormal basis $\{e_i\}_{i \in I}$. Let P be an orthogonal projection of $\mathcal{K}$ onto $\mathcal{H} \subset \mathcal{K}$. Then, $\{Pe_i\}$ is a Parseval frame for $\mathcal{H} = P(\mathcal{K})$.

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Conversely, let $\{f_i\}_{i\in I}$ be a Parseval frame for $\mathcal{H}$. Then, there exists a larger Hilbert space $\mathcal{K} \supset \mathcal{H}$ with orthonormal basis $\{e_i\}_{i\in I}$ such that $P(e_i) = f_i$, where P is an orthogonal projection of $\mathcal{K}$ onto $\mathcal{H}$.

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- $\mathcal{K}$ can be identified with $\ell^2(I)$.

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Proposition

Let $\mathcal{K}$ be a Hilbert space with orthonormal basis $\{e_i\}_{i \in I}$ and $0 < A \leq B < \infty$. If E is a positive operator on $\mathcal{K}$ with

$$\{A, B\} \subseteq \sigma(E) \subseteq \{0\} \cup [A, B], \tag{1}$$

then $\{Ee_i\}$ is a frame for $\mathcal{H} = E(\mathcal{K})$ with optimal bounds A^2 and B^2.

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- $\mathcal{K}$ can be identified with $\ell^2(I)$.
- E is unitarily equivalent with $S^{1/2} \oplus \mathbf{0}$,
  S frame operator on $\mathcal{H}$, $\mathbf{0}$ acts on $\mathcal{H}^\perp \subset \mathcal{K}$.

Reformulation of problem

Theorem (Antezana, Massey, Ruiz, Stojanoff (2007))

Let $0 < A \leq B < \infty$ and S be a positive operator on a Hilbert space $\mathcal{H}$ with $\sigma(S) \subset [A, B]$. The following sets are equal:

$$\left\{ \{\|f_i\|^2\}_{i\in I} \;\middle|\; \begin{array}{l} \{f_i\}_{i\in I} \text{ is a frame for } \mathcal{H} \text{ with} \\ \text{frame operator } S \end{array} \right\}$$

$$\left\{ \{\langle Ee_i, e_i\rangle\}_{i\in I} \;\middle|\; \begin{array}{c} E \text{ is self-adjoint on } \ell^2(I) \text{ and} \\ \text{unitarily equivalent with } S \oplus \mathbf{0} \end{array} \right\}$$

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Reformulated (Schur-Horn) Problem. Characterize diagonals $\{\langle Ee_i, e_i\rangle\}_{i\in I}$ of a self-adjoint operator E, where $\{e_i\}_{i\in I}$ is any orthonormal basis of $\mathcal{H}$.

Parseval Frames

Definition

A sequence $\{f_i\}_{i \in I}$ *in a Hilbert space* $\mathcal{H}$ *is a tight frame (Parseval frame if* $B = 1$*) if*

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Reformulated Problem. Characterize diagonals of orthogonal projections.
This problem was solved by Kadison (2002) and independently in $\mathbb{R}^N$ and $\mathbb{C}^N$ case by Casazza, Fickus, Kovačevíc, Leon, and Tremain (2006) using frame potentials.

$$\max_{i=1,\ldots,M} ||f_i||^2 \leq \frac{1}{N}\sum_{i=1}^{M} ||f_i||^2 = B.$$

Pythagorean Theorem and Carpenter's Theorem

Theorem (Kadison (2002))

Let $\{d_i\}_{i\in I}$ *be a sequence in* $[0,1]$ *and* $\alpha \in (0,1)$. *Define*

$$C(\alpha) = \sum_{d_i<\alpha} d_i, \qquad D(\alpha) = \sum_{d_i\geq\alpha} (1-d_i).$$

There exists an orthogonal projection on $\ell^2(I)$ *with diagonal* $\{d_i\}_{i\in I} \iff$ *either:*

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2. $C(\alpha), D(\alpha) < \infty$, *and* $C(\alpha) - D(\alpha) \in \mathbb{Z}$.

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- The same condition characterizes sequences of norms of Parseval frames.
- The finite case is a consequence of the Schur-Horn theorem—the necessary and sufficient condition is $\sum d_i \in \mathbb{N}$.

Schur-Horn Theorem

Theorem (Schur (1923), Horn (1954))

Suppose S is an $N \times N$ Hermitian matrix with eigenvalues $\{\lambda_i\}_{i=1}^N$ and diagonal $\{d_i\}_{i=1}^N$ listed in nonincreasing order. Then,

$$\sum_{i=1}^{n} d_i \leq \sum_{i=1}^{n} \lambda_i \quad \forall\, n = 1, \ldots, N \tag{2}$$

with equality when $n = N$.

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- (2) is equivalent to the convexity condition $(d_1, \ldots, d_N) \in \operatorname{conv}\{(\lambda_{\sigma(1)}, \ldots, \lambda_{\sigma(N)}) : \sigma \in S_N\} \subset \mathbb{R}^N$.

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- This is a special case of the Kostant convexity theorem for connected semi-simple groups G when $G = SU(N)$.

Locally invertible positive operators

Theorem (Bownik, Jasper (2011))

Let $0 < A < B < \infty$ *and* $\{d_i\}$ *be a nonsummable sequence in* $[0, B]$. *Define the numbers*

$$C = \sum_{d_i < A} d_i \quad \text{and} \quad D = \sum_{d_i \geq A} (B - d_i).$$

There is a positive operator E *with* $\{A, B\} \subseteq \sigma(E) \subseteq \{0\} \cup [A, B]$ *and diagonal* $\{d_i\}$ $\iff$ *either:*

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1. $C = \infty$ *or* $D = \infty$, *or*
2. $C, D < \infty$ *and* $\exists\, N \in \mathbb{N} \cup \{0\}$,

$$NA \leq C \leq A + B(N - 1) + D.$$

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- The nonsummability $\sum d_i = \infty$ is not a true limitation.
- However, the assumption $A < B$ is essential since the tight case $A = B$ corresponds to Kadison's theorem.

Frames with prescribed lower and upper bounds

Corollary (Bownik, Jasper (2011))

Let $0 < A < B < \infty$ and $\{d_i\}$ be a nonsummable sequence in $[0, B]$. Define the numbers

$$C = \sum_{d_i < A} d_i \quad \text{and} \quad D = \sum_{d_i \geq A} (B - d_i).$$

There is a frame with optimal bounds A and B and $d_i = \|f_i\|^2$ $\iff$ either:

1. $C = \infty$ *or* $D = \infty$, *or*
2. $C, D < \infty$ *and* $\exists\ N \in \mathbb{N} \cup \{0\}$,

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Moving diagonal in desirable configuration

Lemma (Moving toward 0-1)

Let $\{d_i\}_{i\in I}$ *be a sequence in* $[0,B]$. *Let* $I_0, I_1 \subset I$ *be two disjoint finite subsets such that* $\max\{d_i : i \in I_0\} \le \min\{d_i : i \in I_1\}$. *Let*

$$0 \le \eta_0 \le \min\left\{ \sum_{i\in I_0} d_i, \sum_{i\in I_1} (B - d_i) \right\}.$$

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(i) There exists a sequence $\{\tilde{d}_i\}_{i\in I}$ in $[0,B]$ satisfying:

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$$0 \le \eta_0 \le \min\left\{ \sum_{i\in I_0} d_i, \sum_{i\in I_1}(B-d_i)\right\}.$$

(i) There exists a sequence $\{\tilde{d}_i\}_{i\in I}$ *in* $[0,B]$ *satisfying:*

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   $\eta_0 + \sum_{i \in I_1} (B - \tilde{d}_i) = \sum_{i \in I_1} (B - d_i)$.

(ii) $\tilde{E}$ *self-adjoint operator with diagonal* $\{\tilde{d}_i\}_{i \in I} \implies$ *there exists an operator* E *unitarily equivalent to* $\tilde{E}$ *with diagonal* $\{d_i\}_{i \in I}$.

Schur-Horn for operators with 3 point spectrum

Theorem (Jasper (2011))

Let $0 < A < B < \infty$ *and* $\{d_i\}_{i \in I}$ *be a sequence in* $[0, B]$ *with* $\sum d_i = \sum (B - d_i) = \infty$. *Define*

$$C = \sum_{d_i < A} d_i \quad \text{and} \quad D = \sum_{d_i \geq A} (B - d_i).$$

There is a self-adjoint operator E *with diagonal* $\{d_i\}_{i \in I}$ *and* $\sigma(E) = \{0, A, B\} \iff$ *either:*

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2. $C, D < \infty$ *and* $\exists\, N \in \mathbb{N}$, $k \in \mathbb{Z}$

$$C - D = NA + kB \quad \text{and} \quad C \geq (N + k)A.$$

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$\sum d_i = \sum (B - d_i) = \infty$ is not a true limitation.

Frames with prescribed norms and 2 point spectrum

Corollary (Jasper (2011))

Let $0 < A < B < \infty$ *and* $\{d_i\}_{i \in I}$ *be a sequence in* $[0, B]$ *with* $\sum d_i = \sum (B - d_i) = \infty$. *Define*

$$C = \sum_{d_i < A} d_i \quad \text{and} \quad D = \sum_{d_i \geq A} (B - d_i).$$

There is a frame such that $\sigma(S) = \{A, B\}$ *and* $d_i = \|f_i\|^2 \iff$ *either:*

1. $C = \infty$ *or* $D = \infty$, *or*
2. $C, D < \infty$ *and* $\exists\, N \in \mathbb{N}$, $k \in \mathbb{Z}$

$$C - D = NA + kB \quad \text{and} \quad C \geq (N + k)A.$$

Definition

Let $0 < A < B < \infty$ *and* $\{d_i\}_{i \in I}$ *in* $[0, B]$. *Define the sets*

$$I_1 = \{i \in I : d_i < A\}, \; I_2 = \{i \in I : d_i \geq A\},$$

$$J_2 = \{i \in I_2 : d_i < (A+B)/2\}, \; J_3 = I_2 \setminus J_2$$

and the constants (each possibly infinite)

$$C = \sum_{i \in I_1} d_i \qquad D = \sum_{i \in I_2} (B - d_i)$$

$$C_1 = \sum_{i \in I_1} (A - d_i), \; C_2 = \sum_{i \in J_2} (d_i - A), \; C_3 = \sum_{i \in J_3} (B - d_i).$$

Let E *be a bounded operator on a Hilbert space.*
For $\lambda \in \mathbb{C}$ *define* $m_E(\lambda) = \dim \ker(E - \lambda)$.

3 point spectrum and prescribed multiplicites

Theorem (Jasper (2011))

E has diagonal $\{d_i\}$ and $\sigma(E) = \{0, A, B\}$ $\iff$

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E has diagonal $\{d_i\}$ and $\sigma(E) = \{0, A, B\}$ $\iff$

	$m_E(0)$	$m_E(A)$	$m_E(B)$	Condition
(a)	Z	N	K	$\|I\| = Z + N + K$ $\sum_{i\in I} d_i = NA + KB,\ C \ge (N + K - \|I_2\|)A$
(b)	∞	N	K	$\|I_1\| = \infty,$ $\sum_{i\in I} d_i = NA + KB,\ C \ge (N + K - \|I_2\|)A$
(c)	∞	N	∞	$C + D = \infty$ *or* $C, D < \infty,\ \|I_1\| = \|I_2\| = \infty,$ $\exists\, k \in \mathbb{Z}\ C - D = NA + kB,\ C \ge A(N + k)$
(d)	Z	∞	K	$\|I\| = \infty,\ C_1 \le AZ$ $\sum_{i\in I} (d_i - A) = K(B - A) - ZA$
(e)	Z	∞	∞	$C_1 \le AZ,\ C_2 + C_3 = \infty$ *or* $\|I_1 \cup J_2\| = \|J_3\| = \infty,\ C_1 \le AZ,\ C_2, C_3 < \infty$ $\exists\, k \in \mathbb{Z},\ \ C_1 - C_2 + C_3 = (Z - k)A + kB$
(f)	∞	∞	∞	$C + D = \infty$

n point spectrum - Riemann majorization

Definition

Let $0 = A_0 < A_1 < \ldots < A_{n+1} = B$, $n \in \mathbb{N}$.
Let $\{\lambda_i\}_{i \in \mathbb{Z}}$ *be a* **nondecreasing** *sequence which takes values in* $\{A_0, A_1, \ldots, A_{n+1}\}$, *each at least once.*

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Let $\{d_i\}_{i\in\mathbb{Z}}$ be a **nondecreasing** *sequence in $[0, B]$ such that $\sum_{i=-\infty}^{0} d_i < \infty$.*
We say that $\{d_i\}$ satisfies Riemann majorization by $\{A_j\}_{j=0}^{n+1}$ if there exists such a sequence $\{\lambda_i\}_{i\in\mathbb{Z}}$ as above, so that the following two hold:

$$\delta_m := \sum_{i=-\infty}^{m} (d_i - \lambda_i) \geq 0 \qquad \text{for all } m \in \mathbb{Z},$$

$$\lim_{m\to\infty} \delta_m = 0.$$

Definition

Let $0 = A_0 < A_1 < \ldots < A_{n+1} = B$, $n \in \mathbb{N}$.

n point spectrum - Lebesgue majorization

Definition

Let $0 = A_0 < A_1 < \ldots < A_{n+1} = B$, $n \in \mathbb{N}$.
Let $\{d_i\}_{i\in\mathbb{Z}}$ *be* **any** *sequence in* $[0, B]$. *For* $\alpha \in (0, B)$ *define*
$C(\alpha) = \sum_{d_i<\alpha} d_i$ *and* $D(\alpha) = \sum_{d_i\geq\alpha}(B - d_i)$.

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We say that $\{d_i\}$ *satisfies Lebesgue majorization by* $\{A_j\}_{j=0}^{n+1}$ *if*
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$$(B-A_r)C(A_r)+A_rD(A_r) \geq (B-A_r)\sum_{j=1}^{r} A_jN_j+A_r \sum_{j=r+1}^{n}(B-A_j)N_j$$

for all $r = 1, \ldots, n$.

Equivalence of Riemann and Lebesgue majorizations

Theorem

Let $\{d_i\}_{i\in\mathbb{Z}}$ be a nondecreasing sequence in $[0, B]$. Then, $\{d_i\}$ satisfies Riemann majorization by $\{A_j\}_{j=0}^{n+1}$ $\iff$ $\{d_i\}$ satisfies Lebesgue majorization by $\{A_j\}_{j=0}^{n+1}$.

Theorem (Bownik, Jasper (2012))

Let $0 = A_0 < A_1 < \ldots < A_{n+1} = B$, $n \in \mathbb{N}$. *Let* $\{d_i\}_{i \in I} \subset [0, B]$. *Assume* $\sum d_i = \sum (B - d_i) = \infty$.
There exists a self-adjoint operator E *with diagonal* $\{d_i\}_{i \in I}$ *and* $\sigma(E) = \{A_0, A_1, \ldots, A_{n+1}\} \iff$ *either:*

Schur-Horn for operators with finite point spectrum

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for all $r = 1, \ldots, n$.

Question: Given a fixed sequence $\{d_i\} \subset [0,1]$, for what numbers $0 < A < 1$ does there exist a frame $\{f_i\}$ such that $d_i = ||f_i||^2$ and the spectrum of frame operator $\sigma(S) = \{A, 1\}$?

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Let $\{d_i\}_{i \in \mathbb{N}}$ be a sequence in $[0,1]$ and set

$$\mathcal{A} = \left\{ A \in (0,1) : \exists\, E \text{ with } \sigma(E) = \{0, A, 1\} \text{ and diagonal } \{d_i\} \right\}.$$

Either $\mathcal{A} = (0,1)$ or $\mathcal{A}$ is a finite (possibly empty) set.

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Either $\mathcal{A} = (0,1)$ or $\mathcal{A}$ is a finite (possibly empty) set. Moreover, if $\mathcal{A} = \emptyset$, then $\{d_i\}$ is a diagonal of a projection.

Example

Let $\beta \in (0,1)$ and define the sequence $\{d_i\}_{i\in\mathbb{Z}\setminus\{0\}}$ by

$$d_i = \begin{cases} 1-\beta^i, & i>0 \\ \beta^{|i|} & i<0. \end{cases}$$

Then

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Geometric series example

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$$\mathcal{A} = \begin{cases} \varnothing & 0<\beta<1/3, \\ \{\frac{1}{2}\} & 1/3 \le \beta < \frac{-1+\sqrt{13}}{6} \approx 0.434, \\ & \\ & \end{cases}$$

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Extended example

Example

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Determine the possible pairs of numbers (A_1, A_2) such that there exists a frame $\{f_i\}$ with $d_i = ||f_i||^2$ and the spectrum of frame operator $\sigma(S) = \{A_1, A_2, 1\}$?

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In other words we are interested in the set

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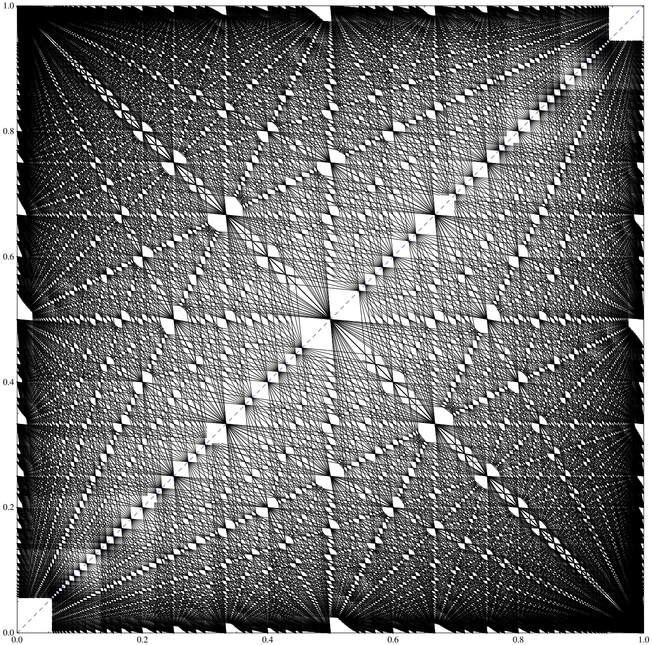
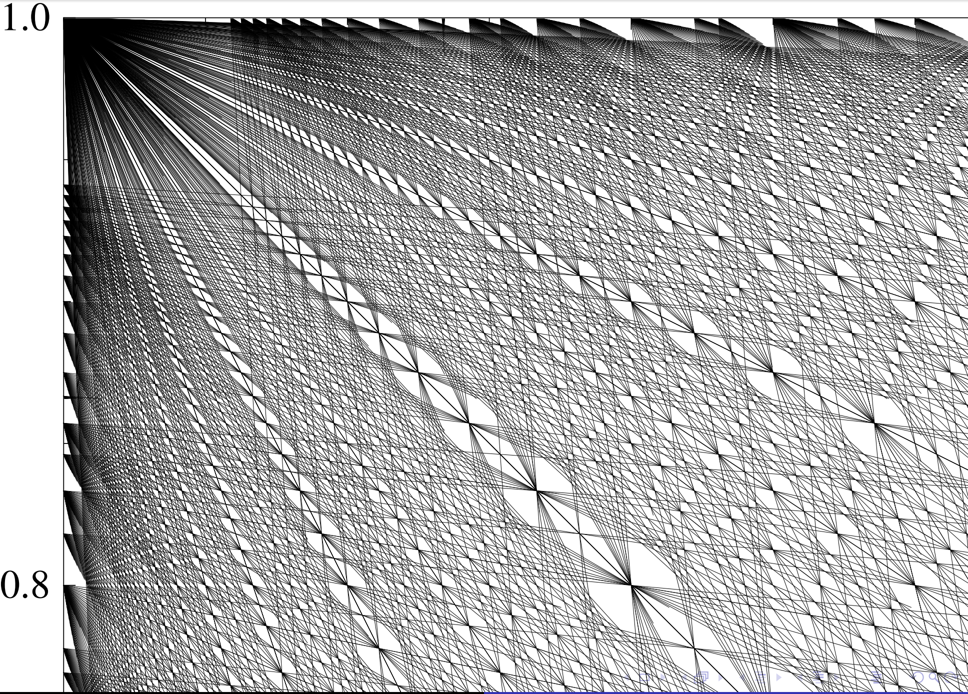
$$\big\{(A_1, A_2) \in (0,1)^2 : \exists\, E \text{ with } \sigma(E) = \{0, A_1, A_2, 1\} \\ \text{and diagonal } \{d_i\}\big\}.$$

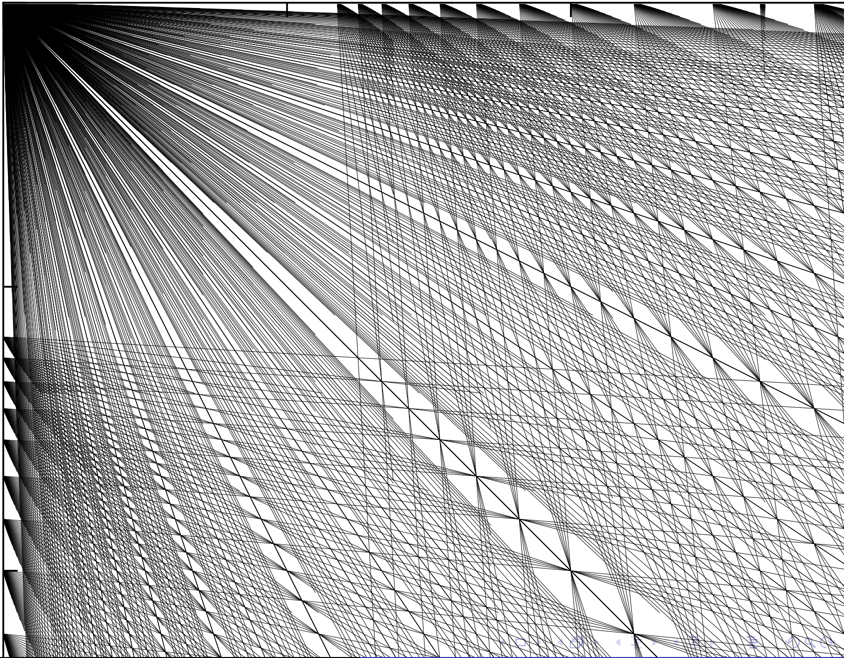
The following picture corresponds to $\beta = 0.8$.

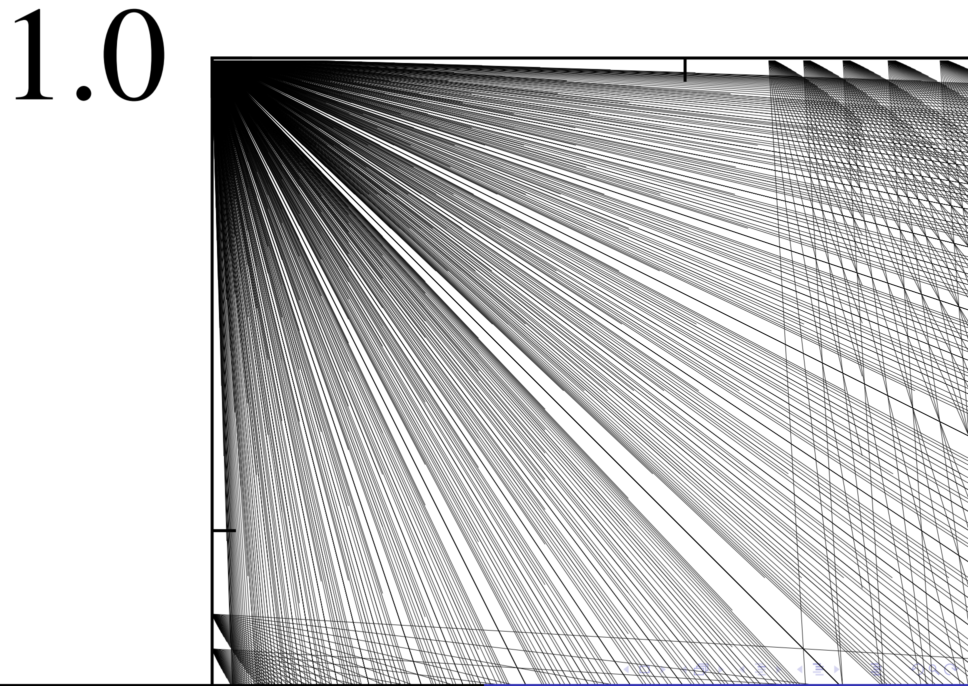


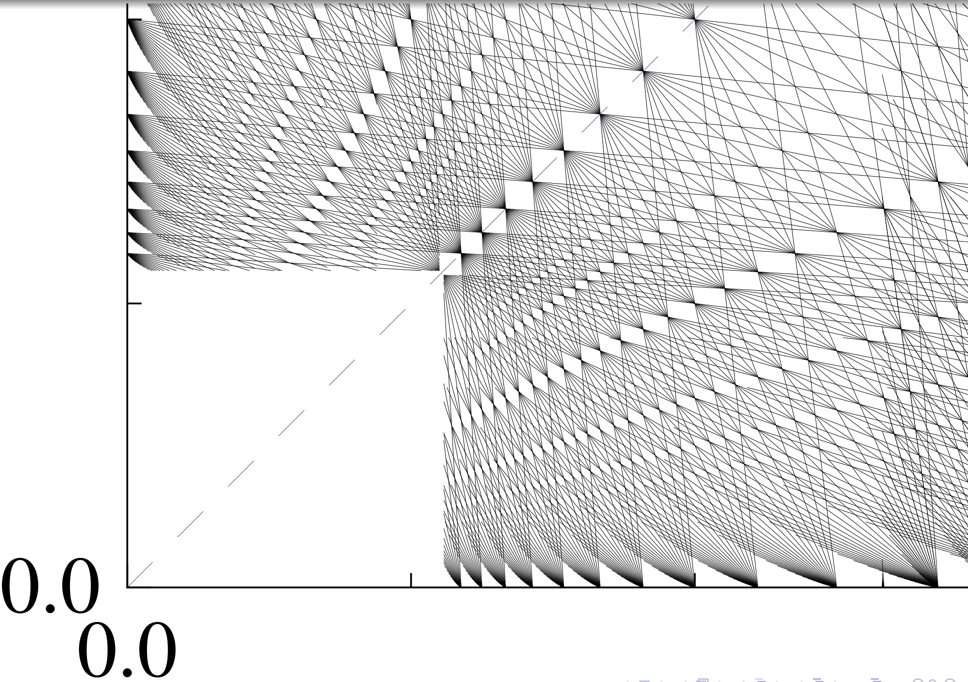
1.0
0.8
0.6
0.4
0.2
0.0
0.0
0.2
0.4
0.6
0.8
1.0

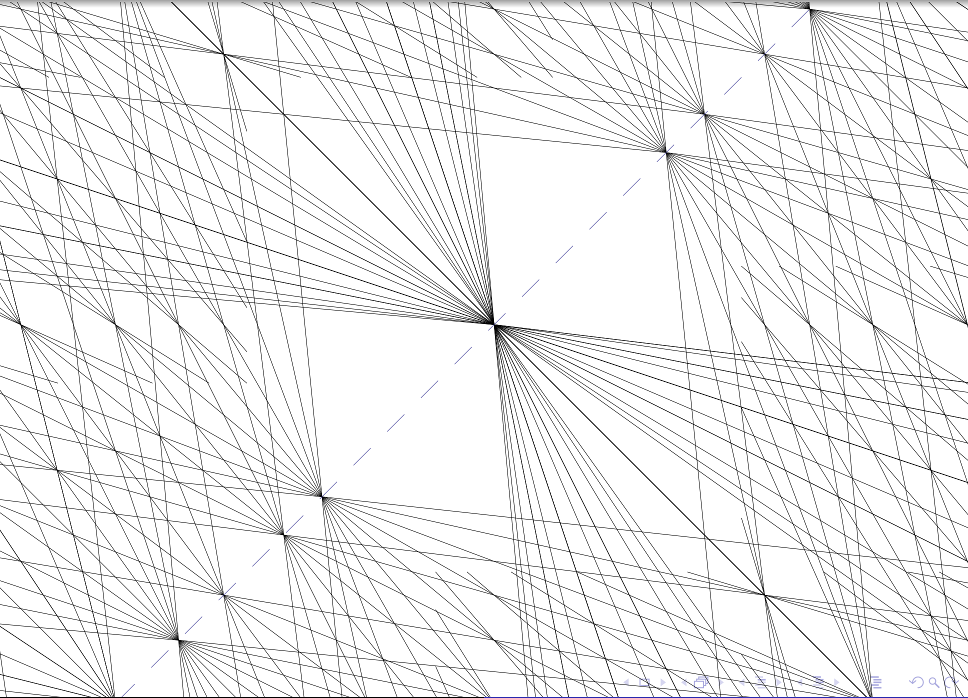

1.0
0.8

1.0




1.0


0.0
0.0



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- Characterize diagonals of operators with finite spectrum and **with prescribed multiplicities**.
- Ultimately extend the Schur-Horn theorem to operators with infinite spectrum beyond the results of Arveson-Kadison (2006) and Kaftal-Weiss (2010).