Transversal Multilinear Harmonic Analysis

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Part 1: A multilinear approach to the Fourier restriction conjecture.

• An introduction to the restriction problem for the Fourier transform.

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- Bilinear and multilinear variants of the restriction problem.

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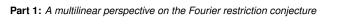
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- Multilinear Radon-like transforms.

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Let S be a smooth compact (d-1)-dimensional submanifold of \mathbb{R}^d , such as the unit sphere \mathbb{S}^{d-1} , section of paraboloid

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For a suitable function $f: \mathbb{R}^d \to \mathbb{C}$ let

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Of course, this operator has a trivial bound,

$$\|\mathcal{R}_{\mathcal{S}}f\|_{L^{\infty}(d\sigma)} \leq \|\widehat{f}\|_{L^{\infty}(\mathbb{R}^d)} \leq \|f\|_{L^{1}(\mathbb{R}^d)}.$$

[Here $d\sigma$ is surface measure on S.]

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In order to understand better the role of curvature it is helpful to instead consider bounds on the *adjoint* restriction operator (or extension operator) \mathcal{R}_S^* given by

$$\mathcal{R}_{\mathcal{S}}^{*}g=\widehat{gd\sigma},$$

where

$$\widehat{gd\sigma}(\xi) = \int_{S} e^{ix\cdot\xi} g(x) d\sigma(x); \quad \xi \in \mathbb{R}^{d}.$$

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Thus, when S is a hyperplane, the only possible $L^p(\mathbb{R}^d)-L^q(d\sigma)$ bound for \mathcal{R}_S^* is the trivial bound

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[We note that more generally, if $S = \{x \in \mathbb{R}^d : x_j = 0\}$ for some $1 \le j \le d$, then $\widehat{gd\sigma} = \widehat{g} \circ \pi_j$ where $\pi_j(\xi) = (\xi_1, \dots, \xi_{j-1}, \xi_{j+1}, \dots, \xi_d)$.]

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Restriction Conjecture (Stein 1960s)

If S has nonvanishing gaussian curvature, $\frac{1}{q} < \frac{d-1}{2d}$ and $\frac{1}{q} \leq \frac{d-1}{d+1} \frac{1}{p'}$, then there exists a constant $C < \infty$ such that

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The conjecture is generated by testing the inequality

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• Necessity of $q > \frac{2d}{d-1}$. This will become apparent in a moment.

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- Latest progress: Bourgain-Guth 2010, Temur 2011.

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In the 1990s a new perspective was introduced which aimed to **exploit curvature in a more geometric way**: this is the so-called *bilinear approach*...

Let $\{S_{\alpha}\}$ be a partition of S by "caps" (or "patches") and write

$$g=\sum_{lpha}g_{lpha}, \;\;\; ext{where} \;\;\; g_{lpha}=g\chi_{\mathcal{S}_{lpha}}.$$

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Now, since S is *curved*, a generic pair $S_{\alpha_1}, S_{\alpha_2}$ will be *transversal* in the sense that if $v_{\alpha_1}, v_{\alpha_2}$ are unit normal vectors to $S_{\alpha_1}, S_{\alpha_2}$ respectively, then $|v_{\alpha_1} \wedge v_{\alpha_2}|$ is bounded below.

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Now, since S is *curved*, a generic pair S_{α_1} , S_{α_2} will be *transversal* in the sense that if v_{α_1} , v_{α_2} are unit normal vectors to S_{α_1} , S_{α_2} respectively, then $|v_{\alpha_1} \wedge v_{\alpha_2}|$ is bounded below.

In order to understand $\|\widehat{g}d\sigma\|_q$ it would thus seem appropriate to study $\|\widehat{g_1}d\sigma_1\widehat{g_2}d\sigma_2\|_{q/2}$ where $d\sigma_1$, $d\sigma_2$ are smooth densities on transversal submanifolds S_1 , S_2 respectively.

Bilinear restriction Conjecture (Tao-Vargas-Vega 1998)

If S_1 , S_2 are <u>transversal</u> and have <u>nonvanishing gaussian curvature</u>, $\frac{1}{q} < \frac{d-1}{2d}$, $\frac{1}{q} \leq \frac{d}{d+2} \frac{1}{p'}$ and $\frac{1}{q} \leq \frac{d-2}{d+2} \frac{1}{p'} + \frac{1}{d+2}$, then there exists a constant $C < \infty$ such that

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- d > 2: Bourgain, Tao-Vargas-Vega (1998), Tao (2003).
- This progress should not be viewed in isolation interlaced with progress in closely related settings, such as that of the cone: Wolff (2001), Tao (2001).

A simple yet revealing example: the bilinear problem for d = 2

Proposition

If S_1 and S_2 are transversal curves in \mathbb{R}^2 then

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• The corresponding *linear* estimate $\|\widehat{gd\sigma}\|_{L^4(\mathbb{R}^2)} \lesssim \|g\|_{L^2(d\sigma)}$ is false.

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- This bilinear estimate is the *endpoint* in the d = 2 bilinear restriction problem;

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- The corresponding *linear* estimate $\|\widehat{gd\sigma}\|_{L^4(\mathbb{R}^2)}\lesssim \|g\|_{L^2(d\sigma)}$ is false.
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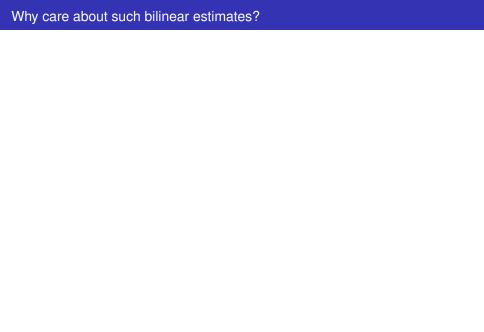
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$$\|(g_1d\sigma_1)*(g_2d\sigma_2)\|_{L^{\infty}(\mathbb{R}^2)} \lesssim \|g_1\|_{L^{\infty}(d\sigma_1)}\|g_2\|_{L^{\infty}(d\sigma_2)}.$$

However,
$$\|(g_1d\sigma_1)*(g_2d\sigma_2)\|_{L^{\infty}(\mathbb{R}^2)} \leq \|g_1\|_{\infty}\|g_2\|_{\infty}\|d\sigma_1*d\sigma_2\|_{L^{\infty}(\mathbb{R}^2)}.$$

Exercise: $d\sigma_1 * d\sigma_2 \in L^{\infty}(\mathbb{R}^2)$ if and only if S_1 and S_2 are transversal.

- The corresponding *linear* estimate $\|\widehat{gd\sigma}\|_{L^4(\mathbb{R}^2)}\lesssim \|g\|_{L^2(d\sigma)}$ is false.
- This bilinear estimate is the *endpoint* in the d = 2 bilinear restriction problem; there is a missing endpoint for d > 2.
- For this estimate (and hence the whole d = 2 bilinear restriction problem), the curvature hypothesis is redundant!



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Despite this great advantage, the bilinear formulation has one main drawback: for d>2 the roles of curvature and transversality are mixed up and difficult to distinguish...

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Let's reflect further on our partition of ${\mathcal S}$ into "caps" (or "patches") $\{{\mathcal S}_\alpha\}$.

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Let $2 \le k \le d$. A k-tuple S_1, \ldots, S_k is *transversal* if there is a constant c > 0 such that

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As may be expected, the case k = d is rather special...

d-linear Restriction Conjecture

If S_1, \ldots, S_d are <u>transversal</u>, $\frac{1}{q} \leq \frac{d-1}{2d}$ and $\frac{1}{q} \leq \frac{d-1}{d} \frac{1}{p'}$, then there exists a constant $C < \infty$ such that

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Key features:

• The *d*-linear conjecture includes (and is equivalent to) the *endpoint* estimate

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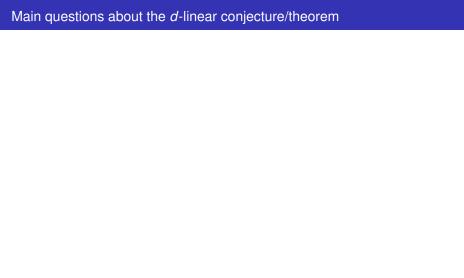
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Theorem (B-Carbery-Tao 2006)

Under the above conditions, given any $\epsilon>0$ there exists a constant $C_\epsilon<\infty$ such that

$$\|\widehat{g_1d\sigma_1}\cdots\widehat{g_dd\sigma_d}\|_{L^{q/d}(B(0,R))} \leq C_{\epsilon}R^{\epsilon}\|g_1\|_{L^p(S_1)}\cdots\|g_d\|_{L^p(S_d)}$$

for all R.



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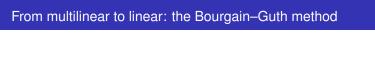
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 - -We'll address this in Part 2 (see the next lecture).



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$$\begin{split} |\widehat{gd\sigma}|^2 &= \Big| \sum_{\alpha_1,\alpha_2} \widehat{g_{\alpha_1}d\sigma} \overline{\widehat{g_{\alpha_2}d\sigma}} \Big| \\ &\leq \sum_{\alpha_1,\alpha_2} |\widehat{g_{\alpha_1}d\sigma} \widehat{g_{\alpha_2}d\sigma}| \end{split}$$

From multilinear to linear: the Bourgain-Guth method

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Thus for every $\xi \in \mathbb{R}^d$,

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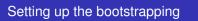
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This suggests a bootstrapping argument....



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Similarly, we appear to need to know how the conjectured bilinear inequality scales...

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Lemma (Linear scaling)

$$\|\widehat{g_{\alpha}d\sigma}\|_{L^{q}(\mathcal{B}(0,R))}\lesssim \mathcal{C}K^{\frac{d+1}{q}-\frac{d-1}{p'}}\|g_{\alpha}\|_{L^{p}(d\sigma)}.$$

Similarly, we appear to need to know how the conjectured bilinear inequality scales...

Lemma (Bilinear scaling)

Suppose that the conjectured bilinear restriction inequality holds with exponents p, q. If $\operatorname{dist}(S_{\alpha_1}, S_{\alpha_2}) \gtrsim \frac{1}{K}$ then

$$\|\widehat{g_{\alpha_1}}\overline{d\sigma}\widehat{g_{\alpha_2}}\overline{d\sigma}\|_{\frac{q}{2}}\lesssim K^{\cdots}\|g_{\alpha_1}\|_{L^p(d\sigma)}\|g_{\alpha_2}\|_{L^p(d\sigma)}.$$

$$\|\widehat{gd\sigma}\|_q^q \lesssim K^{2(d-1)(\frac{q}{2}-1)} \sum_{\mathrm{dist}(S_{\alpha_1},S_{\alpha_2}) \gtrsim 1/K} \|\widehat{g_{\alpha_1}d\sigma}\widehat{g_{\alpha_2}d\sigma}\|_{\frac{q}{2}}^{\frac{q}{2}} + K^{(d-1)(\frac{q}{2}-1)} \sum_{\alpha} \|\widehat{g_{\alpha}d\sigma}\|_q^q,$$

$$\|\widehat{gd\sigma}\|_q^q \lesssim K^{2(d-1)(\frac{q}{2}-1)} \sum_{\mathrm{dist}(S_{\alpha_1},S_{\alpha_2}) \gtrsim 1/K} \|\widehat{g_{\alpha_1}d\sigma}\widehat{g_{\alpha_2}d\sigma}\|_{\frac{q}{2}}^{\frac{q}{2}} + K^{(d-1)(\frac{q}{2}-1)} \sum_{\alpha} \|\widehat{g_{\alpha}d\sigma}\|_q^q,$$

using the bilinear restriction conjecture and the scaling estimates we obtain

$$\|\widehat{\mathit{gd}\sigma}\|_q^q \lesssim \mathit{K}^{\dots} \sum_{\alpha_1,\alpha_2} \|g_{\alpha_1}\|_{\rho}^{\frac{q}{2}} \|g_{\alpha_2}\|_{\rho}^{\frac{q}{2}} +$$

$$\|\widehat{gd\sigma}\|_q^q \lesssim K^{2(d-1)(\frac{q}{2}-1)} \sum_{\operatorname{dist}(S_{\alpha_1},S_{\alpha\alpha}) \geq 1/K} \|\widehat{g_{\alpha_1}d\sigma}\widehat{g_{\alpha_2}d\sigma}\|_{\frac{q}{2}}^{\frac{q}{2}} + K^{(d-1)(\frac{q}{2}-1)} \sum_{\alpha} \|\widehat{g_{\alpha}d\sigma}\|_q^q,$$

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$$\gamma_1(p,q) = \begin{cases} \frac{d+1}{q} - \frac{d-1}{2} & \text{if } p \ge q \\ (d-1)(\frac{1}{2} - \frac{1}{q}) + \frac{d+1}{q} - \frac{d-1}{p'} & \text{if } p < q \end{cases}$$

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If p,q are such that $\gamma_1(p,q)<0$ then by taking K sufficiently large, we obtain $\mathcal{C}(R)<\infty$ uniformly in $R\gg 1$.

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In particular, if $p \ge q$ and $q > \frac{2(d+1)}{d-1}$ (the Stein–Tomas exponent) we obtain the desired linear restriction estimate.



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• The range of exponents p, q for which a linear estimate followed was that for which $\gamma_1(p,q) < 0$.

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Proposition ("Bourgain-Guth" 2010)

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Observation: Integrating *this* in $\xi \in \mathbb{R}^d$ gives

$$C < c_2 K^{power} + c_1 C K^{\frac{2d}{q} - (d-1)}$$

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This generates (the interior of) the full conjectured range of exponents for the linear conjecture.

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Remark. The above proposition is an abstract statement about finite sequences, namely

$$||a||_{\ell^1(\mathbb{Z}_N)} \lesssim N\Big(\sum_{i\neq k} |a_j a_k|^{\frac{q}{2}}\Big)^{\frac{1}{q}} + ||a||_{\ell^q(\mathbb{Z}_N)}.$$

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Next time: Linear estimates from multilinear estimates...

In the last lecture we used an easy version of the Bourgain–Guth method to show that the bilinear conjecture implied the linear conjecture (up to the sharp line).

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In particular, we proved:

Proposition ("Bourgain-Guth" 2010)

$$|\widehat{gd\sigma}(\xi)|^q \lesssim K^{2(d-1)q} \sum_{\mathrm{dist}(\mathcal{S}_{\alpha_1},\mathcal{S}_{\alpha_2}) \gtrsim 1/K} |\widehat{g_{\alpha_1}d\sigma}(\xi)\widehat{g_{\alpha_2}d\sigma}(\xi)|^{\frac{q}{2}} + \sum_{\alpha} |\widehat{g_{\alpha}d\sigma}(\xi)|^q.$$

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Defining C to be the best constant in the inequality

$$\|\widehat{gd\sigma}\|_{L^q(B(0,R))} \le C\|g\|_{L^p(d\sigma)}$$

over all surfaces S (which are "uniformly of elliptic type") of diameter at most 1, we deduced that

$$C \leq c_2 K^{power} + c_1 C K^{\frac{2d}{q} - (d-1)}$$

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Let us see how this approach may be extended in order to deduce linear estimates from *multilinear* ones...

We mimic the bilinear approach and look for a suitable pointwise bound of the form

$$|\widehat{\textit{gd}\sigma}(\xi)|^q \lesssim \textit{K}^{\textit{power}} \sum_{S_{\alpha_1},S_{\alpha_2},S_{\alpha_3} \text{ transversal}} |\widehat{\textit{g}_{\alpha_1}\textit{d}\sigma}(\xi)\widehat{\textit{g}_{\alpha_2}\textit{d}\sigma}(\xi)\widehat{\textit{g}_{\alpha_3}\textit{d}\sigma}(\xi)|^{\frac{q}{3}} + \cdots$$

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For the sake of simplicity let us suppose that $S = \mathbb{S}^{d-1}$, and as before $\{S_{\alpha}\}$ is a partition of S into "caps" of diameter approximately 1/K.

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$$|v_{\alpha_1} \wedge v_{\alpha_2} \wedge v_{\alpha_3}| \gtrsim \frac{1}{K^2}$$

uniformly in the unit normal vectors $v_{\alpha_1}, v_{\alpha_2}, v_{\alpha_3}$ to $S_{\alpha_1}, S_{\alpha_2}, S_{\alpha_3}$ respectively.

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• Observe that if S_{α_1} , S_{α_2} , S_{α_3} are *not* transversal, then they all lie within a distance O(1/K) of some great circle.

(I) there exist $\alpha_1, \alpha_2, \alpha_3$ with $S_{\alpha_1}, S_{\alpha_2}, S_{\alpha_3}$ transversal, such that

$$|\widehat{g_{\alpha_1}d\sigma}(\xi)|,|\widehat{g_{\alpha_2}d\sigma}(\xi)|,|\widehat{g_{\alpha_3}d\sigma}(\xi)|\geq K^{-(d-1)}\max_{\alpha}|\widehat{g_{\alpha}d\sigma}(\xi)|,$$

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(II) there exists a great circle $E \subset S$ such that whenever $\operatorname{dist}(S_{\alpha}, E) \gtrsim \frac{1}{K}$,

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and estimate each term separately.

$$\Big| \sum_{\mathrm{dist}(\mathcal{S}_\alpha, \mathcal{E}) \gtrsim \frac{1}{K}} \widehat{g_\alpha d\sigma}(\xi) \Big| \lesssim \max_\alpha |\widehat{g_\alpha d\sigma}(\xi)| \leq \Big(\sum_\alpha |\widehat{g_\alpha d\sigma}(\xi)|^q \Big)^{\frac{1}{q}},$$

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More specifically, we introduce a second large parameter $K' \leq K$ and a partition $\{S'_{\beta}\}$ of S into larger "caps" of diameter approximately 1/K' to obtain:

$$\Big|\sum_{\mathrm{dist}(S_{\alpha},E)\gtrsim \frac{1}{K}}\widehat{g_{\alpha}d\sigma}(\xi)\Big|\lesssim \max_{\alpha}|\widehat{g_{\alpha}d\sigma}(\xi)|\leq \Big(\sum_{\alpha}|\widehat{g_{\alpha}d\sigma}(\xi)|^{q}\Big)^{\frac{1}{q}},$$

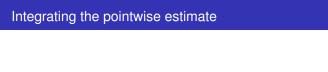
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Proposition (Bourgain-Guth)

$$\begin{split} |\widehat{gd\sigma}|^q &\lesssim \mathcal{K}^{2(d-1)q} \sum_{\substack{S_{\alpha_1}, S_{\alpha_2}, S_{\alpha_3} \text{ transversal}}} |\widehat{g_{\alpha_1}d\sigma}\widehat{g_{\alpha_2}d\sigma}\widehat{g_{\alpha_3}d\sigma}|^{\frac{q}{3}} \\ &+ (\mathcal{K}')^{2(d-2)q} \sum_{\substack{\text{dist}(S'_{\beta_1}, S'_{\beta_2}) \gtrsim \frac{1}{\mathcal{K}'} \\ \text{dist}(S_{\alpha_1}, \mathcal{E}) \lesssim \frac{1}{\mathcal{K}}}} \Big| \Big(\sum_{\substack{\alpha_1: S_{\alpha_1} \subset S'_{\beta_1} \\ \text{dist}(S_{\alpha_1}, \mathcal{E}) \lesssim \frac{1}{\mathcal{K}}}} \widehat{g_{\alpha_1}d\sigma} \Big) \Big(\sum_{\substack{\alpha_2: S_{\alpha_2} \subset S'_{\beta_2} \\ \text{dist}(S_{\alpha_2}, \mathcal{E}) \lesssim \frac{1}{\mathcal{K}}}} \widehat{g_{\alpha_2}d\sigma} \Big) \Big|^{\frac{q}{2}} \\ &+ \sum_{\beta} \Big| \sum_{\alpha: S_{\alpha} \subset S'_{\beta}} \widehat{g_{\alpha}d\sigma} \Big|^{q} + \sum_{\alpha} |\widehat{g_{\alpha}d\sigma}|^{q}. \end{split}$$



Integrating the pointwise estimate

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Let $\{Q\}$ be a tiling of \mathbb{R}^d by cubes of side K, and for each Q assume that $E_{\xi}=E_Q$ for all $\xi\in Q$. Integrating in $\xi\in B(0,R)$ gives,

$$\begin{split} \|\widehat{gd\sigma}\|_q^q &\lesssim K^{2(d-1)q} \sum_{S_{\alpha_1}, S_{\alpha_2}, S_{\alpha_3} \text{ transversal}} \|\widehat{g_{\alpha_1}d\sigma}\widehat{g_{\alpha_2}d\sigma}\widehat{g_{\alpha_3}d\sigma}\|_{\frac{q}{3}}^{\frac{q}{3}} \\ &+ (K')^{2(d-2)q} \sum_{Q} \sum_{\text{dist}(S'_{\beta_1}, S'_{\beta_2}) \gtrsim \frac{1}{K'}} \left\| \Big(\sum_{\substack{\alpha_1: S_{\alpha_1} \subset S'_{\beta_1} \\ \text{dist}(S_{\alpha_1}, E_Q) \lesssim \frac{1}{K}}} \widehat{g_{\alpha_1}d\sigma} \Big) \Big(\sum_{\substack{\alpha_2: S_{\alpha_2} \subset S'_{\beta_2} \\ \text{dist}(S_{\alpha_2}, E_Q) \lesssim \frac{1}{K}}} \widehat{g_{\alpha_2}d\sigma} \Big) \right\|_{L^{\frac{q}{2}}(Q)}^{\frac{q}{2}} \\ &+ \sum_{\beta} \left\| \sum_{\alpha: S_{\alpha_1} \subset S'_{\beta_2}} \widehat{g_{\alpha_2}d\sigma} \right\|_q^q + \sum_{\alpha} \|\widehat{g_{\alpha_2}d\sigma}\|_q^q. \end{split}$$

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$$\begin{split} &\lesssim K^{2(d-1)q} \sum_{S_{\alpha_1}, S_{\alpha_2}, S_{\alpha_3} \text{ transversal}} \|\widehat{g_{\alpha_1}} \widehat{d\sigma} \widehat{g_{\alpha_2}} \widehat{d\sigma} \widehat{g_{\alpha_3}} \widehat{d\sigma} \|_{\frac{q}{3}}^{\frac{q}{3}} \\ &+ (K')^{\textit{power}} \sum_{Q} \sum_{\beta} \left\| \left(\sum_{\substack{\alpha: S_{\alpha} \subset S'_{\beta} \\ \text{dist}(S_{\alpha}, E_Q) \lesssim \frac{1}{K}}} |\widehat{g_{\alpha}} \widehat{d\sigma}|^2 \right)^{\frac{1}{2}} \right\|_{L^q(Q)}^q \\ &+ \sum_{\beta} \left\| \sum_{\alpha: S_{\alpha} \subset S'_{\beta}} \widehat{g_{\alpha}} \widehat{d\sigma} \right\|_q^q + \sum_{\alpha} \|\widehat{g_{\alpha}} \widehat{d\sigma}\|_q^q \end{split}$$

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Remarks.

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- Argument generalises to higher dimensions (Bourgain–Guth, Femur).
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Part 2: A closer look at the multilinear restriction conjecture



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Recall from last time:

d-linear Restriction Conjecture

If S_1,\ldots,S_d are <u>transversal</u>, $\frac{1}{q}\leq \frac{d-1}{2d}$ and $\frac{1}{q}\leq \frac{d-1}{d}\frac{1}{p'}$, then there exists a constant $C<\infty$ such that

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Theorem (B-Carbery-Tao 2006)

Under the above conditions, given any $\epsilon>0$ there exists a constant $C_\epsilon<\infty$ such that

$$\|\widehat{g_1d\sigma_1}\cdots\widehat{g_dd\sigma_d}\|_{L^{q/d}(B(0,R))} \leq C_{\epsilon}R^{\epsilon}\|g_1\|_{L^p(S_1)}\cdots\|g_d\|_{L^p(S_d)}$$

for all R.

A slight refinement

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Theorem (B-Carbery-Tao revisited)

If S_1, \ldots, S_d are transversal, then there exists a constants $C, \kappa < \infty$ such that

$$\|\widehat{g_1d\sigma_1}\cdots\widehat{g_dd\sigma_d}\|_{L^{\frac{1}{d-1}}(B(0,R))} \leq C(\log R)^{\kappa}\|g_1\|_{L^2(S_1)}\cdots\|g_d\|_{L^2(S_d)}.$$

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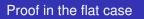
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Just beneath the surface of the conjectured endpoint inequality

$$\|\widehat{g_1d\sigma_1}\cdots\widehat{g_dd\sigma_d}\|_{L^{\frac{2}{d-1}}(\mathbb{R}^d)}\leq C\|g_1\|_{L^2(S_1)}\cdots\|g_d\|_{L^2(S_d)}$$

lies a well-known geometric inequality. Identifying this is the key starting point...



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where $\pi_j(\xi) = (\xi_1, \dots, \xi_{j-1}, \xi_{j+1}, \dots, \xi_d)$.

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$$\|\widehat{g}_1 \circ \pi_1 \cdots \widehat{g}_d \circ \pi_d\|_{L^{\frac{2}{d-1}}(\mathbb{R}^d)} \le C \|g_1\|_{L^2(S_1)} \cdots \|g_d\|_{L^2(S_d)}.$$

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Notice also that

$$|\Omega| \leq |\pi_1(\Omega)|^{\frac{1}{d-1}} \cdots |\pi_d(\Omega)|^{\frac{1}{d-1}} \iff |\Omega| \geq \frac{|\Omega|}{|\pi_1(\Omega)|} \cdots \frac{|\Omega|}{|\pi_d(\Omega)|}.$$

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Proof of this special case of the d-linear restriction conjecture is very rigid - does not extend routinely to general transversal S_1, \ldots, S_d . However, an important aspect of it does: we may indeed reduce the general case to a **positive** inequality of Loomis—Whitney type...

Claim: the desired inequality

$$\|\widehat{g_{1}d\sigma_{1}}\cdots\widehat{g_{d}d\sigma_{d}}\|_{L^{\frac{1}{d-1}}(B(0,R))} \leq C(\log R)^{\kappa}\|g_{1}\|_{L^{2}(S_{1})}\cdots\|g_{d}\|_{L^{2}(S_{d})}$$

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where $T(\alpha_j) = \pi_{\alpha_j}^{-1} B(\alpha_j)$ is a doubly infinite cylindrical tube in \mathbb{R}^d of width $\sim \delta$ and direction $\ker \pi_{\alpha_j}$.

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$$\left\| \prod_{j=1}^d \Bigl(\sum_{\alpha_j \in \mathcal{A}_j} \chi_{T(\alpha_j)} \Bigr) \right\|_{L^{\frac{1}{d-1}}(\mathbb{R}^d)}^{\frac{1}{d-1}} \lesssim \prod_{j=1}^d \left(\delta^{d-1} \# \mathcal{A}_j \right)^{\frac{1}{d-1}}.$$

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may be reduced to a certain "vector" Loomis-Whitney inequality, namely

$$\int_{\mathbb{R}^d} \prod_{j=1}^d \left(\sum_{\alpha_j \in \mathcal{A}_j} f_{\alpha_j} \circ \pi_{\alpha_j} \right)^{\frac{1}{d-1}} \lesssim \prod_{j=1}^d \left(\sum_{\alpha_j \in \mathcal{A}_j} \int_{\mathbb{R}^{d-1}} f_{\alpha_j} \right)^{\frac{1}{d-1}},$$

where for each j, A_j is an indexing set and π_{α_j} is a linear map which is sufficiently close to the fixed π_j (the jth coordinate hyperplane projection).

This is an equivalent functional form of a Kakeya-type inequality – if we set $f_{\alpha_j} = \chi_{B(\alpha_j)}$, where $B(\alpha_j)$ denotes a δ -ball in \mathbb{R}^{d-1} , then

$$f_{\alpha_j} \circ \pi_{\alpha_j} = \chi_{T(\alpha_j)},$$

where $T(\alpha_j) = \pi_{\alpha_j}^{-1} B(\alpha_j)$ is a doubly infinite cylindrical tube in \mathbb{R}^d of width $\sim \delta$ and direction $\ker \pi_{\alpha_j}$. Thus the above inequality becomes

$$\left\| \prod_{j=1}^d \left(\sum_{\alpha_j \in \mathcal{A}_j} \chi_{\mathcal{T}(\alpha_j)} \right) \right\|_{L^{\frac{1}{d-1}}(\mathbb{R}^d)}^{\frac{1}{d-1}} \lesssim \prod_{j=1}^d \left(\delta^{d-1} \# \mathcal{A}_j \right)^{\frac{1}{d-1}}.$$

This is the endpoint case of the so-called *d-linear Kakeya theorem*:

Let $\mathbb{T}_1,\ldots,\mathbb{T}_d$ be families of doubly-infinite δ -tubes. If these families are transversal and $q\geq \frac{d}{d-1}$ then

$$\bigg\| \prod_{j=1}^d \Big(\sum_{T_j \in \mathbb{T}_j} \chi_{T_j} \Big) \bigg\|_{L^{q/d}(\mathbb{R}^d)} \lesssim \prod_{j=1}^d \delta^{q/d} \# \mathbb{T}_j.$$

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Proposition (B-Carbery-Tao revisited)

There exists a constant $c \ge 1$ independent of R such that

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Remark. Such restriction-Kakeya bootstrapping results originate in work of Bourgain (1991).

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for some constant κ .



The proof of the *d*-linear restriction theorem now boils down to

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Next time we will discuss each of these ingredients in turn....

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Last time we saw that proving the d-linear restriction theorem boils down to

- proving the inductive proposition; i.e. $\mathcal{C}_{Rest}(R) \lesssim \mathcal{C}_{Rest}(R^{1/2})\mathcal{C}_{Kak}(R^{-1/2})$, and
- proving the *d*-linear Kakeya theorem; i.e. $C_{\text{Kak}}(\delta) \lesssim 1$.

Recall that $C_{Rest}(R)$ denotes the best constant in the inequality

$$\left\| \prod_{j=1}^{d} \widehat{g_{j}} d\sigma_{j} \right\|_{L^{\frac{2}{d-1}}(B(0,R))} \leq C \prod_{j=1}^{d} \|g_{j}\|_{2}.$$

The endpoint *d*-linear restriction conjecture is thus $C_{Rest}(R) \lesssim 1$; we are aiming for $C_{Rest}(R) \lesssim (\log R)^{\kappa}$ for some $\kappa > 0$.

Similarly, $\mathcal{C}_{Kak}(\delta)$ denotes the best constant in

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We now discuss each of these ingredients in turn....

Why is $\mathcal{C}_{Rest}(R) \lesssim \mathcal{C}_{Rest}(R^{1/2})\mathcal{C}_{Kak}(R^{-1/2})$ true?

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Proposition (B-Carbery-Tao revisited)

There exists a constant $c \ge 1$ independent of $0 < \delta \ll \delta' \ll 1$ such that

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Remark. This proposition captures a certain "self-similarity property" or "invariance property" of the d-linear Kakeya inequality.

Let us prove this carefully...

Proof. For notational simplicity we write $\mathcal{C}_{\mathrm{Kak}}(\delta)=\mathcal{C}(\delta)$ here.

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$$\Big\| \prod_{j=1}^d \Big(\sum_{T_j \in \mathbb{T}_j} \chi_{T_j} \Big) \Big\|_{L^{q/d}(\mathbb{R}^d)}^{q/d} = \sum_Q \Big\| \prod_{j=1}^d \Big(\sum_{T_j \in \mathbb{T}_j} \chi_{T_j} \Big) \Big\|_{L^{q/d}(Q)}^{q/d}$$

$$\begin{split} \left\| \prod_{j=1}^{d} \left(\sum_{T_{j} \in \mathbb{T}_{j}} \chi_{T_{j}} \right) \right\|_{L^{q/d}(\mathbb{R}^{d})}^{q/d} &= \sum_{Q} \left\| \prod_{j=1}^{d} \left(\sum_{T_{j} \in \mathbb{T}_{j}} \chi_{T_{j}} \right) \right\|_{L^{q/d}(Q)}^{q/d} \\ &= \sum_{Q} \left\| \prod_{j=1}^{d} \left(\sum_{T_{j} \in \mathbb{T}_{j}^{Q}} \chi_{T_{j} \cap Q} \right) \right\|_{L^{q/d}(\mathbb{R}^{d})}^{q/d} \end{split}$$

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Tile \mathbb{R}^d by cubes Q of side δ' with $\delta \ll \delta' \leq 1$. Clearly,

$$\begin{split} \left\| \prod_{j=1}^{d} \left(\sum_{T_{j} \in \mathbb{T}_{j}} \chi_{T_{j}} \right) \right\|_{L^{q/d}(\mathbb{R}^{d})}^{q/d} &= \sum_{Q} \left\| \prod_{j=1}^{d} \left(\sum_{T_{j} \in \mathbb{T}_{j}} \chi_{T_{j}} \right) \right\|_{L^{q/d}(Q)}^{q/d} \\ &= \sum_{Q} \left\| \prod_{j=1}^{d} \left(\sum_{T_{j} \in \mathbb{T}_{j}^{Q}} \chi_{T_{j} \cap Q} \right) \right\|_{L^{q/d}(\mathbb{R}^{d})}^{q/d} \\ &\lesssim \mathcal{C}(\delta/\delta') \sum_{Q} \left(\prod_{j=1}^{d} \# \mathbb{T}_{j}^{Q} \right)^{q/d} \\ &\lesssim \mathcal{C}(\delta/\delta') \sum_{Q} \left(\prod_{j=1}^{d} \sum_{T_{j} \in \mathbb{T}_{j}} \chi_{\widetilde{T}_{j}}(x_{Q}) \right)^{q/d} \\ &\lesssim \mathcal{C}(\delta/\delta') \int_{\mathbb{R}^{d}} \left(\prod_{j=1}^{d} \sum_{T_{j} \in \mathbb{T}_{j}} \chi_{\widetilde{T}_{j}} \right)^{q/d} \lesssim \mathcal{C}(\delta/\delta') \mathcal{C}(\delta') \delta^{d} \left(\prod_{j=1}^{d} \# \mathbb{T}_{j} \right)^{q/d}. \end{split}$$

Thus

$$C(\delta) \leq C(\delta/\delta')C(\delta')$$

with implicit constant uniform in $0 < \delta < \delta' < 1$.

OK, but we really wanted to prove:

Proposition

There exists a constant $c \ge 1$ independent of R such that

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Proof as before but with one extra ingredient: a wave packet decomposition

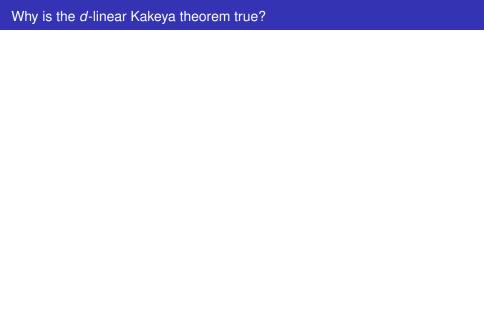
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Idea: Return to the functional form of the *d*-linear Kakeya inequality

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For each α_j let u_{α_j} solve the heat equation $\partial_t u = \Delta u$ with initial data f_{α_j} , and consider the functional

$$Q(t) = \int_{\mathbb{R}^d} \prod_{j=1}^d \left(\sum_{\alpha_j \in \mathcal{A}_j} u_{\alpha_j} \circ \pi_{\alpha_j} \right)^{\frac{1}{d-1}}.$$

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- The *d*-linear Kakeya inequality maybe rewritten as $Q(0) \lesssim \lim_{t \to \infty} Q(t)$. Thus if Q(t) were nondecreasing we'd be done!
- If $\pi_{\alpha_j} = \pi_j$ for all $\alpha_j \in \mathcal{A}_j$ then

$$Q(t) = Q_{LW}(t) := \int_{\mathbb{R}^d} \prod_{j=1}^d u_j(t, \pi_j x)^{\frac{1}{d-1}} dx,$$

where $u_j = \sum_{\alpha_j \in \mathcal{A}_j} u_{\alpha_j}$ and $f_j = \sum_{\alpha_j \in \mathcal{A}_j} f_{\alpha_j}$.



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$$\left(\prod_{i=1}^{d} H_{t} \circ \pi_{j}(x)\right)^{\frac{1}{d-1}} = (4\pi t)^{-\frac{d}{2}} e^{-\frac{|x|^{2}}{4t}}$$

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Proof (BCCT/K. Ball). Observe first that if $H_t(y) = (4\pi t)^{-\frac{d-1}{2}} e^{-\frac{|y|^2}{4t}}$ is the heat kernel on \mathbb{R}^{d-1} then

$$\left(\prod_{j=1}^{d} H_{t} \circ \pi_{j}(x)\right)^{\frac{1}{d-1}} = (4\pi t)^{-\frac{d}{2}} e^{-\frac{|x|^{2}}{4t}}$$

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A suitably robust "calculus" proof of the above special case may be adapted to obtain the following:

Theorem (B-Carbery-Tao 2006)

Let $q>\frac{d}{d-1}$ and $\epsilon>0$. Suppose that

$$\|(\pi_{\alpha_j}^*\pi_{\alpha_j})^{1/2}-(\pi_j^*\pi_j)^{1/2}\|<\epsilon$$

for all $\alpha_j \in \mathcal{A}_j$ and $1 \leq j \leq d$. Then provided ϵ is sufficiently small there exists a weight function $W = W(t, x, (\pi_{\alpha_j})_{j=1}^d, q) = 1 + O(\epsilon)$ for which

$$\widetilde{Q}_q(t) := t^{rac{1}{2}(d-1)(q-rac{d}{d-1})} \int_{\mathbb{R}^d} \prod_{j=1}^d \Bigl(\sum_{lpha_j \in \mathcal{A}_j} u_{lpha_j}(t,\pi_{lpha_j})\Bigr)^{q/d} W$$

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Non-endpoint d-linear Kakeya follows from the subsequent inequality

$$Q_q(\delta^2) \lesssim \widetilde{Q}_q(\delta^2) \leq \widetilde{Q}_q(\infty) \lesssim Q_q(\infty).$$

Jonathan Bennett (Birmingham)

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We can get still more out of bootstrapping...

Curvy d-linear Kakeya.

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Attempts to generalise the linear restriction conjecture to cover such operators have had limited success. The multilinear setting appears to be much better behaved...

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If $\ker d_{\xi}d_{\chi}\Phi_{1}(0),\ldots$, $\ker d_{\xi}d_{\chi}\Phi_{d}(0)$ $\operatorname{span}\mathbb{R}^{d}$ (e.g. $d_{\xi}d_{\chi}\Phi_{j}(0)=\pi_{j}$) then for each $\epsilon>0$ there is a constant C_{ϵ} such that

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The proof is yet another bootstrapping argument:

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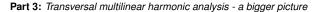
Further applications may be found in:

- J. Bourgain, "Moment inequalities for trigonometric polynomials with spectrum in curved hypersurfaces", preprint 2011.
- J. Bourgain, "On the Schrödinger maximal function in higher dimension", preprint 2012.
- J. Bourgain, P. Shao, C. Sogge, X. Yao, "On LP-resolvent estimates and the density of eigenvalues for compact Riemannian manifolds", preprint 2012.
- S. Lee, A. Vargas, "On the cone multiplier in \mathbb{R}^3 ", JFA 2012.

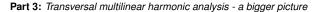
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Rather than impose additional "curvature" conditions on Φ

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It is natural to look for $L^p - L^q$ control of such operators in terms of the large parameter λ , under nondegeneracy conditions on the phase Φ . Starting point:

Theorem (Hörmander)

If d' = d and

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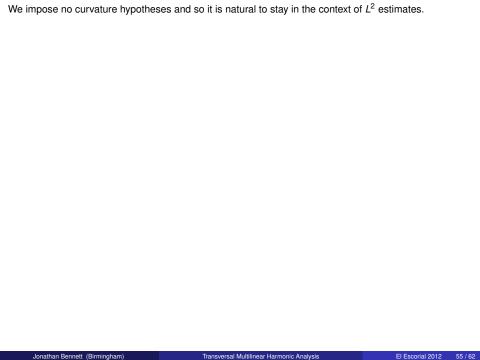
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Rather than impose additional "curvature" conditions on Φ (there is a vast and very important literature on this – see Seeger, El Escorial 2000), let us move to a multilinear setting and impose "transversality" conditions...



$$\int_{\mathbb{R}^d} \prod_{j=1}^k |T_{j,\lambda} f_j|^{2p_j} \leq C \lambda^{-d} \prod_{j=1}^k ||f_j||_{L^2(\mathbb{R}^{d_j})}^{2p_j},$$

where the $T_{j,\lambda}$ are associated to phase functions $\Phi_j: \mathbb{R}^{d_j} \times \mathbb{R}^d \to \mathbb{R}$, and $\mathbf{p} = (\rho_j) \in (0,1]^k$.

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$$\sum_{i=1}^k \rho_j L_j^* L_j = I$$

then BL(L, p) = 1

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Tentative Conjecture (Oscillatory Brascamp-Lieb)

Let (\mathbf{L},\mathbf{p}) be a Brascamp–Lieb datum for which $\mathbf{BL}(\mathbf{L},\mathbf{p})<\infty$, and for each $1\leq j\leq k$ suppose that $\Phi_j:\mathbb{R}^{d_j}\times\mathbb{R}^d\to\mathbb{R}$ is smooth in a neighbourhood of the origin in $\mathbb{R}^{d_j}\times\mathbb{R}^d$ and satisfies $d_\xi d_x \Phi_j(0)=L_j$ for each $1\leq j\leq k$. Then

$$\int_{\mathbb{R}^d} \prod_{j=1}^k |T_{j,\lambda} f_j|^{2p_j} \lesssim \lambda^{-d} \prod_{j=1}^k \|f_j\|_{L^2(\mathbb{R}^{d_j})}^{2p_j}.$$

Tentative Conjecture (Oscillatory Brascamp-Lieb)

Let (\mathbf{L}, \mathbf{p}) be a Brascamp–Lieb datum for which $\mathbf{BL}(\mathbf{L}, \mathbf{p}) < \infty$, and for each $1 \le j \le k$ suppose that $\Phi_j : \mathbb{R}^{d_j} \times \mathbb{R}^d \to \mathbb{R}$ is smooth in a neighbourhood of the origin in $\mathbb{R}^{d_j} \times \mathbb{R}^d$ and satisfies $d_\xi d_x \Phi_j(0) = L_j$ for each $1 \le j \le k$. Then

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Squeezing "as much as possible" out of our bootstrapping arguments we obtain:

Theorem (B-Carbery-Tao revisited)

Suppose (\mathbf{L}, \mathbf{p}) is such that $L_i^* L_j$ is an orthogonal projection for each $1 \le j \le k$ and

$$\sum_{j=1}^k p_j L_j^* L_j = I.$$

If $d_\xi d_X \Phi_j(0) = L_j$ for each $1 \le j \le k$ then given any $\epsilon > 0$ there exists a constant $C_\epsilon < \infty$ such that

$$\int_{\mathbb{R}^d} \prod_{j=1}^k |T_{j,\lambda} f_j|^{2p_j} \leq C_{\epsilon} \lambda^{-d+\epsilon} \prod_{j=1}^k \|f_j\|_{L^2(\mathbb{R}^{d_j})}^{2p_j}.$$

A natural description of a multilinear Radon-like transform is a mapping R of the form

$$Rg(x) = \int_{\mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_m}} g_1(y_1) \cdots g_m(y_m) \delta(F(y, x)) \psi(y, x) dy,$$

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where $g=(g_j)_{j=1}^m, g_j: \mathbb{R}^{d_j} \to \mathbb{C}$ is a suitable test function, $x \in \mathbb{R}^n$ and $F: \mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_m} \times \mathbb{R}^n \to \mathbb{R}^N$ is a suitably smooth function.

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$$\int_{\mathbb{R}^{d_1}\times\cdots\times\mathbb{R}^{d_{m+1}}}\prod_{j=1}^{m+1}g_j(y_j)\delta(F(y))\psi(y)\mathrm{d}y\lesssim \prod_{j=1}^{m+1}\left\|g_j\right\|_{L^{f_j}(\mathbb{R}^{d_j})}.$$

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By parametrising the support of the distribution $\delta \circ F$ we may often write the above inequality in the form

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$$\int_{\mathbb{R}^d} \prod_{j=1}^{m+1} (f_j \circ B_j)^{p_j} \psi \lesssim \prod_{j=1}^{m+1} \left(\int_{\mathbb{R}^{d_j}} f_j \right)^{p_j};$$

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$$\int_{\mathbb{R}^d} \prod_{j=1}^{m+1} g_j(B_j(x)) \psi(x) dx \lesssim \prod_{j=1}^{m+1} \|g_j\|_{L^{r_j}(\mathbb{R}^{d_j})}$$

for some typically nonlinear maps $B_j:\mathbb{R}^d o \mathbb{R}^{d_j}$. Setting $f_j=g_j^{r_j}$ and $p_j=\frac{1}{r_j}$, this becomes

$$\int_{\mathbb{R}^d} \prod_{j=1}^{m+1} (f_j \circ B_j)^{p_j} \psi \lesssim \prod_{j=1}^{m+1} \left(\int_{\mathbb{R}^{d_j}} f_j \right)^{p_j};$$

i.e. a Brascamp–Lieb inequality, but with *nonlinear* maps B_j .

Let (\mathbf{L}, \mathbf{p}) be a Brascamp–Lieb datum for which $\mathbf{BL}(\mathbf{L}, \mathbf{p}) < \infty$, and for each $1 \le j \le k$ let $B_j : \mathbb{R}^d \to \mathbb{R}^{d_j}$ be a smooth submersion in a neighbourhood of a point $x_0 \in \mathbb{R}^d$ with $dB_j(x_0) = L_j$ for each $1 \le j \le k$.

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• This conjecture follows from the Oscillatory Brascamp–Lieb conjecture on specialising to phases of the form $\Phi_i(x,\xi) = \langle x, B_i(\xi) \rangle$.

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Remark. Under the above hypotheses $BL(\mathbf{L}, \mathbf{p}) < \infty$ if and only if $p_1 = \cdots = p_k = \frac{1}{k-1}$, and in which case

$$\mathbf{BL}(\mathbf{L},\mathbf{p}) = \left| \star \bigwedge_{j=1}^{k} \star X_{j}(L_{j}) \right|^{-\frac{1}{k-1}},$$

where $X_j(L_j) \in \Lambda^{d_j}(\mathbb{R}^d)$ denotes the wedge product of the rows of the $d_j \times d$ matrix L_j , and \star the Hodge star.

Corollary (B–Carbery–Wright 2005 (d = 3); B–Bez, 2010 (d > 3))

If $G: (\mathbb{R}^{d-1})^{d-1} \to \mathbb{R}$ is a smooth function such that

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• This is a convolution-type Radon-like transform estimate; i.e. of the form

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 There are versions with symmetric hypotheses on F (non-convolution type, naturally requiring exterior-algebraic formulations): B–Bez–Gutiérrez 2012.

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If Suppose S_1,\dots,S_d are transversal, $1\leq q\leq \infty$ and $p'\leq (d-1)q'$, then

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- Such inequalities have been successfully applied to the well-posedness of the Zakharov system (Bejenaru–Herr–Holmer–Tataru, Bejenaru–Herr 2010/11), handling certain "transverse interaction terms" in certain bilinear $X_{s,b}$ estimates.

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