Transversal Multilinear Harmonic Analysis

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Part 1: *A multilinear approach to the Fourier restriction conjecture.*

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The classical Fourier restriction conjecture

Let S be a smooth compact $(d-1)$-dimensional submanifold of $\mathbb{R}^d$, such as the unit sphere $\mathbb{S}^{d-1}$, section of paraboloid

$$\{x = (x', x_d) : x_d = |x'|^2\}$$

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Of course, this operator has a trivial bound,

$$\|\mathcal{R}_S f\|_{L^\infty(d\sigma)} \leq \|\widehat{f}\|_{L^\infty(\mathbb{R}^d)} \leq \|f\|_{L^1(\mathbb{R}^d)}.$$

[Here $d\sigma$ is surface measure on S.]

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In order to understand better the role of curvature it is helpful to instead consider bounds on the *adjoint restriction operator* (or *extension operator*) $\mathcal{R}_S^*$ given by

$$\mathcal{R}_S^* g = \widehat{g d\sigma},$$

where

$$\widehat{g d\sigma}(\xi) = \int_S e^{ix\cdot\xi} g(x) d\sigma(x); \quad \xi \in \mathbb{R}^d.$$

Example. If S is a hyperplane, for example $S = \{x \in \mathbb{R}^d : x_d = 0\}$, then

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Thus, when S is a hyperplane, the only possible $L^p(\mathbb{R}^d) - L^q(d\sigma)$ bound for $\mathcal{R}_S^*$ is the trivial bound

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[We note that more generally, if $S = \{x \in \mathbb{R}^d : x_j = 0\}$ for some $1 \leq j \leq d$, then $\widehat{gd\sigma} = \widehat{g} \circ \pi_j$ where $\pi_j(\xi) = (\xi_1, \ldots, \xi_{j-1}, \xi_{j+1}, \ldots, \xi_d)$.]

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If S has nonvanishing gaussian curvature, $\frac{1}{q} < \frac{d-1}{2d}$ and $\frac{1}{q} \leq \frac{d-1}{d+1}\frac{1}{p'}$, then there exists a constant $C < \infty$ such that

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- **Necessity of $q > \frac{2d}{d-1}$.**
  This will become apparent in a moment.

Progress on the restriction conjecture

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- Subsequent progress for $d > 2$:
  Wolff 1995,
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- Latest progress: Bourgain–Guth 2010, Temur 2011.

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In the 1990s a new perspective was introduced which aimed to **exploit curvature in a more geometric way**: this is the so-called *bilinear approach*...

A more geometric approach to exploiting curvature: the emergence of transversality considerations

Let $\{S_\alpha\}$ be a partition of S by “caps" (or “patches") and write

$$g = \sum_\alpha g_\alpha, \quad \text{where} \quad g_\alpha = g\chi_{S_\alpha}.$$

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Now, since S is *curved*, a generic pair $S_{\alpha_1}, S_{\alpha_2}$ will be *transversal* in the sense that if $v_{\alpha_1}, v_{\alpha_2}$ are unit normal vectors to $S_{\alpha_1}, S_{\alpha_2}$ respectively, then $|v_{\alpha_1} \wedge v_{\alpha_2}|$ is bounded below.

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In order to understand $\|\widehat{gd\sigma}\|_q$ *it would thus seem appropriate to study* $\|\widehat{g_1 d\sigma_1} \widehat{g_2 d\sigma_2}\|_{q/2}$ *where* $d\sigma_1$*,* $d\sigma_2$ *are smooth densities on transversal submanifolds* S_1*,* S_2 *respectively.*

Bilinear restriction Conjecture (Tao–Vargas–Vega 1998)

If S_1, S_2 are transversal and have nonvanishing gaussian curvature, $\frac{1}{q} < \frac{d-1}{2d}$, $\frac{1}{q} \leq \frac{d}{d+2}\frac{1}{p'}$ *and* $\frac{1}{q} \leq \frac{d-2}{d+2}\frac{1}{p'} + \frac{1}{d+2}$, *then there exists a constant* $C < \infty$ *such that*

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The bilinear restriction conjecture

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- $d > 2$: Bourgain, Tao–Vargas–Vega (1998), Tao (2003).
- This progress should not be viewed in isolation - interlaced with progress in closely related settings, such as that of the cone: Wolff (2001), Tao (2001).

A simple yet revealing example: the bilinear problem for $d = 2$

Proposition

If S_1 and S_2 are transversal curves in $\mathbb{R}^2$ then

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- The corresponding *linear* estimate $\|\widehat{gd\sigma}\|_{L^4(\mathbb{R}^2)} \lesssim \|g\|_{L^2(d\sigma)}$ is false.
- This bilinear estimate is the *endpoint* in the $d = 2$ bilinear restriction problem;

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Exercise: $d\sigma_1 * d\sigma_2 \in L^\infty(\mathbb{R}^2)$ if and only if S_1 and S_2 are transversal.

Observations:

- The corresponding *linear* estimate $\|\widehat{gd\sigma}\|_{L^4(\mathbb{R}^2)} \lesssim \|g\|_{L^2(d\sigma)}$ is false.
- This bilinear estimate is the *endpoint* in the $d = 2$ bilinear restriction problem; there is a missing endpoint for $d > 2$.

A simple yet revealing example: the bilinear problem for $d = 2$

Proposition

If S_1 and S_2 are transversal curves in $\mathbb{R}^2$ then

$$\|\widehat{g_1 d\sigma_1}\,\widehat{g_2 d\sigma_2}\|_{L^2(\mathbb{R}^2)} \lesssim \|g_1\|_{L^2(d\sigma_1)}\|g_2\|_{L^2(d\sigma_2)}.$$

By Plancherel's theorem the proposition is equivalent to

$$\|(g_1 d\sigma_1) * (g_2 d\sigma_2)\|_{L^2(\mathbb{R}^2)} \lesssim \|g_1\|_{L^2(d\sigma_1)}\|g_2\|_{L^2(d\sigma_2)}.$$

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- This bilinear estimate is the *endpoint* in the $d = 2$ bilinear restriction problem; there is a missing endpoint for $d > 2$.
- For this estimate (and hence the whole $d = 2$ bilinear restriction problem), *the curvature hypothesis is redundant!*

Why care about such bilinear estimates?

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Theorem (Tao–Vargas–Vega 1998)

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$$\|\widehat{g_1 d\sigma_1}\,\widehat{g_2 d\sigma_2}\|_{L^{\tilde{q}/2}(\mathbb{R}^d)} \leq C\|g_1\|_{L^{\tilde{p}}(S_1)}\|g_2\|_{L^{\tilde{p}}(S_2)}.$$

holds for all $(\widetilde{p}, \widetilde{q})$ *in a neighbourhood of* (p, q) *then the conjectured linear inequality*

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Despite this great advantage, the bilinear formulation has one main drawback: for $d > 2$ **the roles of curvature and transversality are mixed up and difficult to distinguish...**

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Of course more is true: If $k \leq d$, and S is *curved* then a generic *k-tuple* $S_{\alpha_1}, \ldots, S_{\alpha_k}$ will be *transversal* in the sense that if $v_{\alpha_1}, \ldots, v_{\alpha_k}$ are unit normal vectors to $S_{\alpha_1}, \ldots, S_{\alpha_k}$ respectively, then $|v_{\alpha_1} \wedge \cdots \wedge v_{\alpha_k}|$ is bounded below.

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Let $2 \leq k \leq d$. A k-tuple $S_1, \ldots, S_k$ is *transversal* if there is a constant $c > 0$ such that

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As may be expected, the case $k = d$ is rather special...

The d-linear restriction conjecture

d-linear Restriction Conjecture

If $S_1, \ldots, S_d$ are transversal, $\frac{1}{q} \leq \frac{d-1}{2d}$ and $\frac{1}{q} \leq \frac{d-1}{d}\frac{1}{p'}$, then there exists a constant $C < \infty$ such that

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Theorem (B–Carbery–Tao 2006)

Under the above conditions, given any $\epsilon > 0$ there exists a constant $C_\epsilon < \infty$ such that

$$\|\widehat{g_1 d\sigma_1} \cdots \widehat{g_d d\sigma_d}\|_{L^{q/d}(B(0,R))} \leq C_\epsilon R^\epsilon \|g_1\|_{L^p(S_1)} \cdots \|g_d\|_{L^p(S_d)}$$

for all R.

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- Why do the d-linear estimates appear to be more tractable?

  –We'll address this in Part 2 (see the next lecture).

From multilinear to linear: the Bourgain–Guth method

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$$|\widehat{gd\sigma}|^2 \leq \sum_{\mathrm{dist}(S_{\alpha_1},S_{\alpha_2})\gtrsim 1/K} |\widehat{g_{\alpha_1}d\sigma}\widehat{g_{\alpha_2}d\sigma}| + \sum_{\mathrm{dist}(S_{\alpha_1},S_{\alpha_2})\lesssim 1/K} |\widehat{g_{\alpha_1}d\sigma}\widehat{g_{\alpha_2}d\sigma}|$$

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Thus for every $\xi \in \mathbb{R}^d$,

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This suggests a bootstrapping argument....

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Fix $R \gg 1$. Let $\mathcal{C} = \mathcal{C}(R)$ denote the best constant in the inequality

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We need to know how this inequality scales...

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over all surfaces S which are *of diameter at most* 1 *and the graph of an elliptic phase function* (that is, “close to the base of the paraboloid").
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Lemma (Bilinear scaling)

Suppose that the conjectured bilinear restriction inequality holds with exponents p, q. If $\mathrm{dist}(S_{\alpha_1}, S_{\alpha_2}) \gtrsim \frac{1}{K}$ *then*

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In particular, if $p \geq q$ and $q > \frac{2(d+1)}{d-1}$ (the Stein–Tomas exponent) we obtain the desired linear restriction estimate.

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This generates (the interior of) the full conjectured range of exponents for the linear conjecture.

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$$|\widehat{gd\sigma}(\xi)|^q \lesssim K^{2(d-1)q} \sum_{\mathrm{dist}(S_{\alpha_1},S_{\alpha_2})\gtrsim 1/K} |\widehat{g_{\alpha_1}d\sigma}(\xi)\widehat{g_{\alpha_2}d\sigma}(\xi)|^{\frac{q}{2}} + \sum_{\alpha} |\widehat{g_\alpha d\sigma}(\xi)|^q.$$

Proof. For a given $\xi \in \mathbb{R}^d$ either

(I) there exist α_1, α_2 with $\mathrm{dist}(S_{\alpha_1}, S_{\alpha_2}) \gtrsim \frac{1}{K}$ such that

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If (I) then

$$\begin{aligned}|\widehat{gd\sigma}(\xi)| \leq \sum_\alpha |\widehat{g_\alpha d\sigma}(\xi)| &\lesssim K^{d-1} \max_\alpha |\widehat{g_\alpha d\sigma}(\xi)| \\ &\leq K^{2(d-1)} |\widehat{g_{\alpha_1(\xi)}d\sigma}(\xi)|^{\frac{1}{2}} |\widehat{g_{\alpha_2(\xi)}d\sigma}(\xi)|^{\frac{1}{2}} \\ &\leq K^{2(d-1)} \Big(\sum_{\mathrm{dist}(S_{\alpha_1},S_{\alpha_2})\gtrsim 1/K} |\widehat{g_{\alpha_1}d\sigma}(\xi)\widehat{g_{\alpha_2}d\sigma}(\xi)|^{\frac{q}{2}} \Big)^{\frac{1}{q}}.\end{aligned}$$

Recall: (II) There exists α_0 such that whenever $\mathrm{dist}(S_{\alpha_0}, S_\alpha) \gtrsim \frac{1}{K}$,

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Remark. The above proposition is an abstract statement about finite sequences, namely

$$\|a\|_{\ell^1(\mathbb{Z}_N)} \lesssim N\Big(\sum_{j\neq k} |a_j a_k|^{\frac{q}{2}}\Big)^{\frac{1}{q}} + \|a\|_{\ell^q(\mathbb{Z}_N)}.$$

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Next time: Linear estimates from *multilinear* estimates...

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$$\|\widehat{gd\sigma}\|_{L^q(B(0,R))} \leq C\|g\|_{L^p(d\sigma)}$$

over all surfaces S (which are "uniformly of elliptic type") of diameter at most 1, we deduced that

$$\mathcal{C} \leq c_2 K^{power} + c_1 \mathcal{C} K^{\frac{2d}{q}-(d-1)}$$

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Let us see how this approach may be extended in order to deduce linear estimates from *multilinear* ones...

Trilinear to linear in 3 dimensions (Bourgain–Guth proper)

We mimic the bilinear approach and look for a suitable pointwise bound of the form

$$|\widehat{gd\sigma}(\xi)|^q \lesssim K^{power} \sum_{S_{\alpha_1}, S_{\alpha_2}, S_{\alpha_3} \text{ transversal}} |\widehat{g_{\alpha_1} d\sigma}(\xi)\widehat{g_{\alpha_2} d\sigma}(\xi)\widehat{g_{\alpha_3} d\sigma}(\xi)|^{\frac{q}{3}} + \cdots$$

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- We will say that three caps $S_{\alpha_1}, S_{\alpha_2}, S_{\alpha_3}$ are *transversal* if

$$|v_{\alpha_1} \wedge v_{\alpha_2} \wedge v_{\alpha_3}| \gtrsim \frac{1}{K^2}$$

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  uniformly in the unit normal vectors $v_{\alpha_1}, v_{\alpha_2}, v_{\alpha_3}$ to $S_{\alpha_1}, S_{\alpha_2}, S_{\alpha_3}$ respectively.
- Observe that if $S_{\alpha_1}, S_{\alpha_2}, S_{\alpha_3}$ are *not* transversal, then they all lie within a distance $O(1/K)$ of some great circle.

For a given $\xi \in \mathbb{R}^d$ either

(I) there exist $\alpha_1, \alpha_2, \alpha_3$ with $S_{\alpha_1}, S_{\alpha_2}, S_{\alpha_3}$ transversal, such that

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and estimate each term separately.

By definition of E, the second term satisfies

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More specifically, we introduce a second large parameter $K' \leq K$ and a partition $\{S'_\beta\}$ of S into larger "caps" of diameter approximately $1/K'$ to obtain:

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Proposition (Bourgain–Guth)

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$$+(K')^{2(d-2)q}\sum_{\mathrm{dist}(S'_{\beta_1},S'_{\beta_2})\gtrsim\frac{1}{K'}}\Big|\Big(\sum_{\substack{\alpha_1:S_{\alpha_1}\subset S'_{\beta_1}\\ \mathrm{dist}(S_{\alpha_1},E)\lesssim\frac{1}{K}}}\widehat{g_{\alpha_1}d\sigma}\Big)\Big(\sum_{\substack{\alpha_2:S_{\alpha_2}\subset S'_{\beta_2}\\ \mathrm{dist}(S_{\alpha_2},E)\lesssim\frac{1}{K}}}\widehat{g_{\alpha_2}d\sigma}\Big)\Big|^{\frac{q}{2}}$$

$$+\sum_\beta\Big|\sum_{\alpha:S_\alpha\subset S'_\beta}\widehat{g_\alpha d\sigma}\Big|^q+\sum_\alpha|\widehat{g_\alpha d\sigma}|^q.$$

Integrating the pointwise estimate

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$$+ (K')^{power} \sum_Q \sum_{\text{dist}(S'_{\beta_1}, S'_{\beta_2}) \gtrsim \frac{1}{K'}} \Bigg\| \Bigg(\sum_{\substack{\alpha_1 : S_{\alpha_1} \subset S'_{\beta_1} \\ \text{dist}(S_{\alpha_1}, E_Q) \lesssim \frac{1}{K}}} |\widehat{g_{\alpha_1} d\sigma}|^2 \Bigg)^{\frac{1}{2}} \Bigg(\sum_{\substack{\alpha_2 : S_{\alpha_2} \subset S'_{\beta_2} \\ \text{dist}(S_{\alpha_2}, E_Q) \lesssim \frac{1}{K}}} |\widehat{g_{\alpha_2} d\sigma}|^2 \Bigg)^{\frac{1}{2}} \Bigg\|_{L^{\frac{q}{2}}(Q)}^{\frac{q}{2}}$$

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$$+\sum_\beta\Big\|\sum_{\alpha:S_\alpha\subset S'_\beta}\widehat{g_\alpha d\sigma}\Big\|_q^q+\sum_\alpha\|\widehat{g_\alpha d\sigma}\|_q^q.$$

(Use of a bilinear Córdoba estimate.)

$$\lesssim K^{2(d-1)q} \sum_{S_{\alpha_1}, S_{\alpha_2}, S_{\alpha_3} \text{ transversal}} \| \widehat{g_{\alpha_1} d\sigma} \widehat{g_{\alpha_2} d\sigma} \widehat{g_{\alpha_3} d\sigma} \|_{\frac{q}{3}}^{\frac{q}{3}}$$

$$+ (K')^{power} \sum_Q \sum_\beta \Big\| \Big(\sum_{\substack{\alpha : S_\alpha \subset S'_\beta \\ \mathrm{dist}(S_\alpha, E_Q) \lesssim \frac{1}{K}}} |\widehat{g_\alpha d\sigma}|^2 \Big)^{\frac{1}{2}} \Big\|_{L^q(Q)}^q$$

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$$\lesssim K^{2(d-1)q} \sum_{S_{\alpha_1}, S_{\alpha_2}, S_{\alpha_3} \text{ transversal}} \|\widehat{g_{\alpha_1} d\sigma}\widehat{g_{\alpha_2} d\sigma}\widehat{g_{\alpha_3} d\sigma}\|_{\frac{q}{3}}^{\frac{q}{3}}$$

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$$= K^{2(d-1)q} \sum_{S_{\alpha_1}, S_{\alpha_2}, S_{\alpha_3} \text{ transversal}} \|\widehat{g_{\alpha_1} d\sigma}\widehat{g_{\alpha_2} d\sigma}\widehat{g_{\alpha_3} d\sigma}\|_{\frac{q}{3}}^{\frac{q}{3}}$$

$$+ (K')^{power} \Big(\frac{K}{K'} \Big)^{\frac{q}{2}-1} \sum_Q \sum_\beta \sum_{\substack{\alpha: S_\alpha \subset S'_\beta \\ \text{dist}(S_\alpha, E_Q) \lesssim \frac{1}{K}}} \|\widehat{g_\alpha d\sigma}\|_{L^q(Q)}^q$$

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$$
\begin{aligned}
&\lesssim K^{2(d-1)q} \sum_{S_{\alpha_1}, S_{\alpha_2}, S_{\alpha_3} \text{ transversal}} \| \widehat{g_{\alpha_1} d\sigma} \widehat{g_{\alpha_2} d\sigma} \widehat{g_{\alpha_3} d\sigma} \|_{\frac{q}{3}}^{\frac{q}{3}} \\
&+ (K')^{power} K^{\frac{q}{2}-1} \sum_{\alpha} \| \widehat{g_\alpha d\sigma} \|_q^q \\
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Using the trilinear restriction conjecture and the scaling estimate we obtain constants c_1, c_2, c_3 such that

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whenever $p \geq q$.

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Using the trilinear restriction conjecture and the scaling estimate we obtain constants c_1, c_2, c_3 such that

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Recall from last time:

d-linear Restriction Conjecture

If $S_1, \ldots, S_d$ are transversal, $\frac{1}{q} \leq \frac{d-1}{2d}$ and $\frac{1}{q} \leq \frac{d-1}{d}\frac{1}{p'}$, then there exists a constant $C < \infty$ such that

$$\|\widehat{g_1 d\sigma_1} \cdots \widehat{g_d d\sigma_d}\|_{L^{q/d}(\mathbb{R}^d)} \leq C\|g_1\|_{L^p(S_1)} \cdots \|g_d\|_{L^p(S_d)}.$$

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Recall that

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Theorem (B–Carbery–Tao 2006)

Under the above conditions, given any $\epsilon > 0$ there exists a constant $C_\epsilon < \infty$ such that

$$\|\widehat{g_1 d\sigma_1} \cdots \widehat{g_d d\sigma_d}\|_{L^{q/d}(B(0,R))} \leq C_\epsilon R^\epsilon \|g_1\|_{L^p(S_1)} \cdots \|g_d\|_{L^p(S_d)}$$

for all R.

A slight refinement

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Our current goal is to explain why this d-linear problem appears so tractable. We sketch a proof of the following slight refinement of the previous d-linear restriction theorem:

Theorem (B–Carbery–Tao revisited)

If $S_1, \ldots, S_d$ are <u>*transversal*</u>*, then there exists a constants $C, \kappa < \infty$ such that*

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Just beneath the surface of the conjectured endpoint inequality

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lies a well-known geometric inequality. Identifying this is the key starting point...

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This is the *Loomis–Whitney inequality* with a suboptimal constant.

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Notice also that

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Proof of this special case of the d-linear restriction conjecture is very rigid - does not extend routinely to general transversal $S_1, \ldots, S_d$.

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Proof of this special case of the d-linear restriction conjecture is very rigid - does not extend routinely to general transversal $S_1, \ldots, S_d$. However, an important aspect of it does: we may indeed reduce the general case to a **positive** inequality of Loomis–Whitney type...

The general (non-flat) case

Claim: the desired inequality

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This is the endpoint case of the so-called *d-linear Kakeya theorem*:

Theorem (d-linear Kakeya; B–Carbery–Tao/Guth)

Let $\mathbb{T}_1, \ldots, \mathbb{T}_d$ be families of doubly-infinite δ-tubes. If these families are transversal and $q \geq \frac{d}{d-1}$ then

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Remark. Such restriction–Kakeya bootstrapping results originate in work of Bourgain (1991).

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Next time we will discuss each of these ingredients in turn....

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We now discuss each of these ingredients in turn....

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Let us prove this carefully...

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Thus

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Observations.

- The d-linear Kakeya inequality maybe rewritten as $Q(0) \lesssim \lim_{t\to\infty} Q(t)$. Thus if $Q(t)$ were nondecreasing we'd be done!

A heat-flow approach to d-linear Kakeya

Idea: Return to the functional form of the d-linear Kakeya inequality

$$\int_{\mathbb{R}^d} \prod_{j=1}^{d} \Big(\sum_{\alpha_j \in \mathcal{A}_j} f_{\alpha_j} \circ \pi_{\alpha_j} \Big)^{\frac{1}{d-1}} \lesssim \prod_{j=1}^{d} \Big(\sum_{\alpha_j \in \mathcal{A}_j} \int_{\mathbb{R}^{d-1}} f_{\alpha_j} \Big)^{\frac{1}{d-1}},$$

and regard the f_{α_j} as initial temperature distributions.
For each α_j let u_{α_j} solve the heat equation $\partial_t u = \Delta u$ with initial data f_{α_j}, and consider the functional

$$Q(t) = \int_{\mathbb{R}^d} \prod_{j=1}^{d} \Big(\sum_{\alpha_j \in \mathcal{A}_j} u_{\alpha_j} \circ \pi_{\alpha_j} \Big)^{\frac{1}{d-1}}.$$

Observations.

- The d-linear Kakeya inequality maybe rewritten as $Q(0) \lesssim \lim_{t\to\infty} Q(t)$. Thus if $Q(t)$ were nondecreasing we'd be done!
- If $\pi_{\alpha_j} = \pi_j$ for all $\alpha_j \in \mathcal{A}_j$ then

$$Q(t) = Q_{LW}(t) := \int_{\mathbb{R}^d} \prod_{j=1}^{d} u_j(t, \pi_j x)^{\frac{1}{d-1}}\, dx,$$

  where $u_j = \sum_{\alpha_j \in \mathcal{A}_j} u_{\alpha_j}$ and $f_j = \sum_{\alpha_j \in \mathcal{A}_j} f_{\alpha_j}$.

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$$\Big(\prod_{j=1}^{d} H_t \circ \pi_j(x)\Big)^{\frac{1}{d-1}} = (4\pi t)^{-\frac{d}{2}} e^{-\frac{|x|^2}{4t}}$$

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$$\begin{aligned} Q_{LW}(t) &= \int_{\mathbb{R}^d} \prod_{j=1}^d (H_t * f_j(\pi_j x))^{\frac{1}{d-1}}\, dx \\ &= \int_{\mathbb{R}^d} \Big(\prod_{j=1}^d H_t * f_j(\pi_j \cdot)\Big)^{\frac{1}{d-1}} * \Big(\prod_{j=1}^d H_{t'}(\pi_j \cdot)\Big)^{\frac{1}{d-1}} \qquad \text{(Fubini)} \\ &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \Big(\prod_{j=1}^d H_t * f_j(\pi_j y) H_{t'}(\pi_j x - \pi_j y)\Big)^{\frac{1}{d-1}}\, dy dx \qquad \text{(linearity of } \pi_j) \end{aligned}$$

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&= \int_{\mathbb{R}^d} \Big(\prod_{j=1}^{d} H_{t'} * H_t * f_j(\pi_j x)\Big)^{\frac{1}{d-1}} dx = Q_{LW}(t + t').
\end{aligned}$$

A suitably robust “calculus" proof of the above special case may be adapted to obtain the following:

Theorem (B–Carbery–Tao 2006)

Let $q > \frac{d}{d-1}$ *and* $\epsilon > 0$. *Suppose that*

$$\|(\pi_{\alpha_j}^* \pi_{\alpha_j})^{1/2} - (\pi_j^* \pi_j)^{1/2}\| < \epsilon$$

for all $\alpha_j \in \mathcal{A}_j$ *and* $1 \leq j \leq d$. *Then provided* ϵ *is sufficiently small there exists a weight function* $W = W(t, x, (\pi_{\alpha_j})_{j=1}^d, q) = 1 + O(\epsilon)$ *for which*

$$\widetilde{Q}_q(t) := t^{\frac{1}{2}(d-1)(q - \frac{d}{d-1})} \int_{\mathbb{R}^d} \prod_{j=1}^{d} \Big(\sum_{\alpha_j \in \mathcal{A}_j} u_{\alpha_j}(t, \pi_{\alpha_j}) \Big)^{q/d} W$$

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Non-endpoint d-linear Kakeya follows from the subsequent inequality

$$Q_q(\delta^2) \lesssim \widetilde{Q}_q(\delta^2) \leq \widetilde{Q}_q(\infty) \lesssim Q_q(\infty).$$

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We can get *still more* out of bootstrapping...

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Remark. If the curves are *algebraic* with bounded degree then $\mathcal{C}_{\mathrm{CurvyKak}}(\delta) \lesssim 1$ (Bourgain–Guth).

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d-linear oscillatory integral operators. If $\Sigma : U(\subset \mathbb{R}^{d-1}) \to \mathbb{R}^d$ is a smooth parametrisation of S, then we may write

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Attempts to generalise the linear restriction conjecture to cover such operators have had limited success. The multilinear setting appears to be much better behaved...

Suppose we have d such operators $T_{\lambda,1}, \ldots, T_{\lambda,d}$ associated with phase functions $\Phi_1, \ldots, \Phi_d : \mathbb{R}^{d-1} \times \mathbb{R}^d \to \mathbb{R}$.

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The proof is *yet another bootstrapping argument*:

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Further applications may be found in:

- J. Bourgain, “Moment inequalities for trigonometric polynomials with spectrum in curved hypersurfaces", preprint 2011.
- J. Bourgain, “On the Schrödinger maximal function in higher dimension", preprint 2012.
- J. Bourgain, P. Shao, C. Sogge, X. Yao, “On L^p-resolvent estimates and the density of eigenvalues for compact Riemannian manifolds", preprint 2012.
- S. Lee, A. Vargas, “On the cone multiplier in $\mathbb{R}^3$", JFA 2012.

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  We denote by $\mathbf{BL}(\mathbf{L}, \mathbf{p})$ the smallest value of C for which (BL) holds (the "Brascamp–Lieb constant"). Important example: if $(\mathbf{L}, \mathbf{p})$ is such that $L_j^* L_j$ is an orthogonal projection and

$$\sum_{j=1}^{k} p_j L_j^* L_j = I$$

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We impose no curvature hypotheses and so it is natural to stay in the context of L^2 estimates. We consider inequalities of the form

$$\int_{\mathbb{R}^d} \prod_{j=1}^{k} |T_{j,\lambda} f_j|^{2p_j} \leq C\lambda^{-d} \prod_{j=1}^{k} \|f_j\|_{L^2(\mathbb{R}^{d_j})}^{2p_j},$$

where the $T_{j,\lambda}$ are associated to phase functions $\Phi_j : \mathbb{R}^{d_j} \times \mathbb{R}^d \to \mathbb{R}$, and $\mathbf{p} = (p_j) \in (0,1]^k$.

- If $k = 1$, $d' = d$, $p = 1$ and $\det(\mathrm{Hess}\Phi) \neq 0$ on $\mathrm{supp}(\psi)$ then this is Hörmander's theorem.
- If $k = d$, $d_j = d-1$, $p_j = \frac{1}{d-1}$ and $\Phi_j(x,\xi) = \langle \xi, \Sigma_j(x)\rangle$ then this is the d-linear restriction conjecture.
- In the special case where the phases Φ_j are nondegenerate bilinear forms $\Phi_j(x,\xi) = \langle x, L_j\xi\rangle$, we have $T_{j,\lambda} f_j = \widehat{f_j} \circ L_j(\lambda \cdot)$. By Plancherel's theorem and scaling the above inequality reduces to

$$\int_{\mathbb{R}^d} \prod_{j=1}^{k} (f_j \circ L_j)^{p_j} \leq C \prod_{j=1}^{k} \left(\int_{\mathbb{R}^{d_j}} f_j \right)^{p_j}; \qquad f_j \in L^1(\mathbb{R}^{d_j}, \mathbb{R}_+). \tag{BL}$$

  This is the classical *Brascamp–Lieb inequality* with datum $(\mathbf{L}, \mathbf{p}) = ((L_j), (p_j))$.
  We denote by $\mathbf{BL}(\mathbf{L}, \mathbf{p})$ the smallest value of C for which (BL) holds (the "Brascamp–Lieb constant"). Important example: if $(\mathbf{L}, \mathbf{p})$ is such that $L_j^* L_j$ is an orthogonal projection and

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  then $\mathbf{BL}(\mathbf{L}, \mathbf{p}) = 1$ (this is the "Geometric Brascamp–Lieb inequality" of Ball/Barthe).

Tentative Conjecture (Oscillatory Brascamp–Lieb)

Let $(\mathbf{L}, \mathbf{p})$ *be a Brascamp–Lieb datum for which* $\mathbf{BL}(\mathbf{L}, \mathbf{p}) < \infty$*, and for each* $1 \leq j \leq k$ *suppose that* $\Phi_j : \mathbb{R}^{d_j} \times \mathbb{R}^d \to \mathbb{R}$ *is smooth in a neighbourhood of the origin in* $\mathbb{R}^{d_j} \times \mathbb{R}^d$ *and satisfies* $d_\xi d_x \Phi_j(0) = L_j$ *for each* $1 \leq j \leq k$*. Then*

$$\int_{\mathbb{R}^d} \prod_{j=1}^{k} |T_{j,\lambda} f_j|^{2p_j} \lesssim \lambda^{-d} \prod_{j=1}^{k} \|f_j\|_{L^2(\mathbb{R}^{d_j})}^{2p_j}.$$

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Squeezing “as much as possible" out of our bootstrapping arguments we obtain:

Theorem (B–Carbery–Tao revisited)

Suppose $(\mathbf{L}, \mathbf{p})$ *is such that* $L_j^* L_j$ *is an orthogonal projection for each* $1 \leq j \leq k$ *and*

$$\sum_{j=1}^{k} p_j L_j^* L_j = I.$$

If $d_\xi d_x \Phi_j(0) = L_j$ *for each* $1 \leq j \leq k$ *then given any* $\epsilon > 0$ *there exists a constant* $C_\epsilon < \infty$ *such that*

$$\int_{\mathbb{R}^d} \prod_{j=1}^{k} |T_{j,\lambda} f_j|^{2p_j} \leq C_\epsilon \lambda^{-d+\epsilon} \prod_{j=1}^{k} \|f_j\|_{L^2(\mathbb{R}^{d_j})}^{2p_j}.$$

A natural description of a *multilinear* Radon-like transform is a mapping R of the form

$$Rg(x) = \int_{\mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_m}} g_1(y_1) \cdots g_m(y_m) \delta(F(y,x)) \psi(y,x) dy,$$

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By parametrising the support of the distribution $\delta \circ F$ we may often write the above inequality in the form

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i.e. a Brascamp–Lieb inequality, but with *nonlinear* maps B_j.

Tentative Conjecture (Nonlinear Brascamp–Lieb)

Let $(\mathbf{L}, \mathbf{p})$ *be a Brascamp–Lieb datum for which* $\mathbf{BL}(\mathbf{L}, \mathbf{p}) < \infty$*, and for each* $1 \leq j \leq k$ *let* $B_j : \mathbb{R}^d \to \mathbb{R}^{d_j}$ *be a smooth submersion in a neighbourhood of a point* $x_0 \in \mathbb{R}^d$ *with* $dB_j(x_0) = L_j$ *for each* $1 \leq j \leq k$*.*

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A nonlinear Brascamp–Lieb theorem...

Theorem (B–Bez 2010)

Conjecture true under the additional assumption that

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Remark. Under the above hypotheses $\mathrm{BL}(\mathbf{L}, \mathbf{p}) < \infty$ if and only if $p_1 = \cdots = p_k = \frac{1}{k-1}$, and in which case

$$\mathbf{BL}(\mathbf{L}, \mathbf{p}) = \left| \star \bigwedge_{j=1}^{k} \star X_j(L_j) \right|^{-\frac{1}{k-1}},$$

where $X_j(L_j) \in \Lambda^{d_j}(\mathbb{R}^d)$ denotes the wedge product of the rows of the $d_j \times d$ matrix L_j, and $\star$ the Hodge star.

A palatable multilinear Radon-like transform corollary...

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Corollary (B–Carbery–Wright 2005 ($d=3$); B–Bez, 2010 ($d>3$))

If $G:(\mathbb{R}^{d-1})^{d-1}\rightarrow\mathbb{R}$ *is a smooth function such that*

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Remarks.

- This is a convolution-type Radon-like transform estimate; i.e. of the form

$$\int_{\mathbb{R}^{d_1}\times\cdots\times\mathbb{R}^{d_{m+1}}}\prod_{j=1}^{m+1}g_j(y_j)\delta(F(y))\psi(y)dy\lesssim\prod_{j=1}^{m+1}\|g_j\|_{L^{r_j}(\mathbb{R}^{d_j})}$$

with $F(y)=(y_d-y_{d-1}-\cdots-y_1,G(y_1,\dots,y_{d-1}))$.

A palatable multilinear Radon-like transform corollary...

Corollary (B–Carbery–Wright 2005 ($d = 3$); B–Bez, 2010 ($d > 3$))

If $G : (\mathbb{R}^{d-1})^{d-1} \to \mathbb{R}$ *is a smooth function such that*

$$|\det(\nabla_{y_1} G(0), \dots, \nabla_{y_{d-1}} G(0))| \geq \varepsilon,$$

then there exists a neighbourhood V *of the origin in* $(\mathbb{R}^{d-1})^{d-1}$*, and a constant* C *such that*

$$\int_V g_1(y_1) \cdots g_{d-1}(y_{d-1}) g_d(y_1 + \cdots + y_{d-1}) \delta(G(y))\, \mathrm{d}y \leq C\varepsilon^{-\frac{1}{d-1}} \prod_{j=1}^{d} \|g_j\|_{(d-1)'}$$

for all nonnegative $g_j \in L^{(d-1)'}(\mathbb{R}^{d-1})$*,* $1 \leq j \leq d$*.*

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- This is a convolution-type Radon-like transform estimate; i.e. of the form

$$\int_{\mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_{m+1}}} \prod_{j=1}^{m+1} g_j(y_j) \delta(F(y)) \psi(y) dy \lesssim \prod_{j=1}^{m+1} \|g_j\|_{L^{r_j}(\mathbb{R}^{d_j})}$$

  with $F(y) = (y_d - y_{d-1} - \cdots - y_1, G(y_1, \dots, y_{d-1}))$.
- There are versions with symmetric hypotheses on F (non-convolution type, naturally requiring exterior-algebraic formulations): B–Bez–Gutiérrez 2012.

Some applications...

Corollary (B–Carbery–Wright 2005 ($d=3$); B–Bez, 2010 ($d>3$))

If Suppose $S_1,\ldots,S_d$ are transversal, $1\leq q\leq\infty$ *and* $p'\leq(d-1)q'$*, then*

$$\|g_1d\sigma_1*\cdots*g_dd\sigma_d\|_{L^q(\mathbb{R}^d)}\lesssim\|g_1\|_{L^p(d\sigma_1)}\cdots\|g_d\|_{L^p(d\sigma_d)}.$$

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Setting $q = 2$ gives the (modest) sharp d-linear restriction estimate

$$\|\widehat{g_1 d\sigma_1} \cdots \widehat{g_d d\sigma_d}\|_{L^2(\mathbb{R}^d)} \lesssim \|g_1\|_{(2d-2)'} \cdots \|g_d\|_{(2d-2)'}.$$

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- The above result generalises to submanifolds $S_1, \ldots, S_m$ ($m \neq d$) of *variable codimension* under a suitable transversality condition (B–Bez–Gutiérrez 2012).

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- Such inequalities have been successfully applied to the well-posedness of the Zakharov system (Bejenaru–Herr–Holmer–Tataru, Bejenaru–Herr 2010/11), handling certain “transverse interaction terms" in certain bilinear $X_{s,b}$ estimates.

The end

The end – thank you!