Estimating the maximal singular integral in terms of the singular integral

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Abstract

We consider the problem of estimating the maximal singular integral T^*f in terms of Tf only. The problem arose when I was working on a variation of the David-Semmes problem, consisting in deriving existence of principal values from L^2 boundedness for certain particular Calderón-Zygmund operators defined on subsets of $\mathbb{R}^n$. Concretely, we shall study the validity of the $L^2(\mathbb{R}^n)$-inequality $\|T^*f\|_2 \leq C\,\|Tf\|_2$ in the context of classical Calderón-Zygmund Theory, namely, for operators defined by a principal value convolution $Tf(x) = p.v. \int f(x-y)\,K(y)\,dy$ where the kernel is of the form $K(x) = \Omega(x)/|x|^n,\ x \in \mathbb{R}^n \setminus \{0\}$, with Ω homogeneous of degree 0, of class C^∞ on the unit sphere and with zero integral there. For example, $\Omega(x)$ may be taken to be $P(x)/|x|^d$ where P is a harmonic homogeneous polynomial of degree d. In that case T is called a higher Riesz transform. It turns out that for an even higher Riesz transform a pointwise inequality stronger than the L^2-estimate holds : $T^*f(x) \leq C\,M(Tf)(x),\ x \in \mathbb{R}^n$. Notice that this is a half the well-known Cotlar's inequality $T^*f(x) \leq C\,(M(Tf)(x) + M(f)(x)),\ x \in \mathbb{R}^n$, which is of no use for us because the dependence on f does not come through Tf only. For odd higher Riesz transforms the above inequality fails (even for the Hilbert transform). One has the weaker result $T^*f(x) \leq C\,M^2(Tf)(x),\ x \in \mathbb{R}^n$, where M^2 is the iteration of the maximal operator.

In joint work with Mateu, Orobitg and Carlos Pérez we have described those T of a given parity for which the L^2-inequality holds. The equivalent condition is stated in terms of the spherical harmonics expansion of Ω and is algebraic. It is also equivalent to the appropriate pointwise estimate mentioned above. In the first two lectures I plan to present the proof of the pointwise inequality for even higher Riesz transforms (and say some words on the odd case) and discuss how one gets necessary conditions for the L^2 estimate in a particularly simple situation. A description on what one has to do in the general case will

be sketched. When considering the sufficient condition a tangential contact with 2 dimensional fluid dynamics (vortex patches) arises. In the third lecture I will deal with the basics of vortex patches, showing how the Cauchy and the Beurling transforms enter the scene and I will mention a boundary regularity result for rotating vortex patches we have recently obtained.