Asymptotic Formulas for Eigenvalues of the Multidimensional Schrödinger Operator

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Summary

We consider the d-dimensional Schrödinger operator $L(q(x))$, defined by the differential expression

$$Lu = -\Delta u + q(x)u$$

in d-dimensional paralellepiped F, with the Dirichlet boundary conditions

$$u|_{\partial F} = 0,$$

where ∂F denotes the boundary of the domain F, $x = (x_1, x_2, ..., x_d) \in R^d$, $d \geq 2$, $\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + ... + \frac{\partial^2}{\partial x_d^2}$ is the Laplace operator in R^d, and $q(x)$ is a real-valued function in $L_2(F)$.

Asymptotic formulas for the resonans and non-resonans eigenvalues of the operator $L(q(x))$ in an arbitrary dimension are obtained. We show that the eigenvalues of the operator $L(q(x))$ in the non-resonance domain are close to the eigenvalues of the unperturbed operator $L(0)$, and the eigenvalues in the resonance domain are close to the eigenvalues of the corresponding one-dimensional Sturm-Liouville operator .

Asymptotic formulas for the eigenvalues of the Polyharmonic operator $H(q(x)) = (-\Delta)^l + q(x), \;\; l > \frac{1}{2}$ are discussed.

SOME UNCERTAINTY PRINCIPLES AND HARMONIC ANALYSIS ON NILPOTENT LIE GROUPS

ALI BAKLOUTI

The present work deals with some uncertainty principles related to the decay of an integrable function and its Fourier transform. The mathematical uncertainty principle, roughly speaking, states that a nonzero function and its Fourier transform cannot both be sharply localized. In the case of the abelian group $G = \mathbb{R}^n$, $n \geq 1$, the Fourier transform is defined by

$$\hat{f}(x) = \int_{\mathbb{R}^n} f(y)e^{-2\pi i\langle x,y\rangle}dy, \ x \in \mathbb{R}^n.$$

Let p, q be such that $2 \leq p, q \leq +\infty$ and f a measurable function on $\mathbb{R}$. The Cowling-Price Theorem [3] and the Hardy Theorem [4] assert that the finiteness of $\|e^{\pi a x^2} f\|_p$ and $\|e^{\pi b x^2}\hat{f}\|_q$ implies that f is zero almost everywhere if $ab > 1$. We prove this result for an arbitrary nilpotent Lie group G extending then earlier cases and the classical Hardy theorem proved recently by Kaniuth and Kumar [5]. More precisely, we prove the following:

Theorem 1: *Let G be a connected, simply connected nilpotent Lie group and f be a measurable function on G. Let $2 \leq p, q \leq +\infty$, and $a, b \in \mathbb{R}_+^*$ such that:*

(i) $\displaystyle\int_G e^{pa\pi\|x\|^2}|f(x)|^p dx < +\infty,$

(ii) $\displaystyle\int_{\mathcal{W}} e^{qb\pi\|\xi\|^2}\|\pi_\xi(f)\|_{HS}^q|Pf(\xi)|d\xi < +\infty,$ *where $\mathcal{W}$ is a suitable cross-section for the generic coadjoint orbits in $\mathfrak{g}^*$. Then $f = 0$ almost everywhere if $ab > 1$.*

Here, the notation $\| \ \|_{HS}$ means the Hilbert-Schmidt norm. In the proof, we use some details from the harmonic analysis and the Plancherel theory on nilpotent homogeneous spaces [1].

The case where $1 \leq p, q \leq +\infty$ is also studied for a restricted class of nilpotent Lie Groups. It concerns the so-called $SNPC$ nilpotent Lie groups. A nilpotent Lie group G is said to be $SNPC$ nilpotent (*Special normal polarization condition*), if it is nilpotent and if there exists in its Lie algebra $\mathfrak{g}$ an abelian ideal $\mathfrak{c}$ such that $[\mathfrak{g}, \mathfrak{c}]$ is one dimensional and that the centralizer $\mathfrak{h}$ of $\mathfrak{c}$ is abelian. So, $SNPC$ nilpotent Lie groups are among Lie groups admitting a normal subgroup which polarizes all generic elements of $\mathfrak{g}^*$. We prove the following [2]:

Key words and phrases. Uncertainty principle; Fourier transform; Hardy theorem; Plancherel formula.

Theorem 2: *Let G be a connected, simply connected nilpotent $SNPC$ Lie group. Let $1 \leq p, q \leq +\infty, a, b \in \mathbb{R}_+^*$ and f a square integrable function on G satisfying hypotheses (i) and (ii) of Theorem 1. Then:*

1) $f = 0$ almost everywhere if $ab \geq 1$ and $1 \leq q \leq 2$.

2) $f = 0$ almost everywhere if $ab > 1$ and $q > 2$.

References

1. A. Baklouti, K. Smaoui and J. Ludwig. Estimate of L^p−Fourier Transform Norm on Nilpotent Lie Groups. Journal of Functional Analysis. **199**, 508-520 (2003).

2. A. Baklouti, N. Ben Salah and K. Smaoui. Some uncertainty principles on Nilpotent Lie Groups. Contemporary Mathematics. To appear.

3. M. G Cowling and J. F Price, Generalizations of Heisenberg's inequality, in Harmonic analysis (eds) G Mauceri, F Ricci and G Weiss(1983) LNM, no, 992 (Berlin: Springer) 443-449.

4. G. H Hardy. A theorem concerning Fourier transforms. J. London Math. Soc. **8**, 227-231 (1933).

5. E. Kaniuth and A. Kumar. Hardy's Theorem for nilpotent Lie groups. Math. Proc. Camb. Phil. Soc. **131**, 487-494 (2001).

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BILINEAR CALDERÓN-VAILLANCOURT-TYPE THEOREMS

ÁRPÁD BÉNYI

In this talk we plan to outline several results and techniques that generate bilinear alternatives of a celebrated theorem of Calderón and Vaillancourt about the L^2 boundedness of linear pseudodifferential operators with symbols having bounded derivatives.

We are interested in studying bilinear pseudodifferential operators of the form

$$T_\sigma(f,g)(x) = \int_{\mathbf{R}^n}\int_{\mathbf{R}^n} \sigma(x,\xi,\eta)\widehat{f}(\xi)\widehat{g}(\eta)e^{ix\cdot(\xi+\eta)}d\xi d\eta$$

where the symbols σ satisfy the differential inequalities

$$|\partial_x^\alpha \partial_\xi^\beta \partial_\eta^\gamma \sigma(x,\xi,\eta)| \leq C_{\alpha\beta\gamma}, \tag{1}$$

for all $(x,\xi,\eta) \in \mathbf{R}^{3n}$ and all multi-indices α, β, and γ.

In contrast with the linear case, there exist bilinear operators with (x-independent) symbols satisfying (1) that are not bounded from $L^p \times L^q$ into L^r for $1/p + 1/q = 1/r, 1 \leq p,q,r < \infty$. However, we can obtain positive results by adding some size conditions on the frequency variables. In joint work with R. Torres (to appear in Math. Res. Lett. 2004), using the concept of *bilinear almost orthogonality* we proved the following:

Theorem 1. *Let T be a pseudodifferential operator whose symbol satisfies*

$$|\partial_x^\alpha \partial_\xi^\beta \partial_\eta^\gamma \sigma(x,\xi,\eta)| \leq C_{\alpha\beta\gamma}, \tag{2}$$

$$\sup_x \int \left(\int |\partial_\eta^\alpha \sigma(x,\xi,\eta)|^2 \, d\xi\right)^{1/2} d\eta \leq C_\alpha \tag{3}$$

and

$$\sup_x \int \left(\int |\partial_\xi^\alpha \sigma(x,\xi,\eta)|^2 \, d\eta\right)^{1/2} d\xi \leq C_\alpha, \tag{4}$$

for all multi-indices α,β,γ. Then T can be extended as a bounded operator from $L^2(\mathbf{R}^n) \times L^2(\mathbf{R}^n)$ into $L^1(\mathbf{R}^n)$.

A result that does not require smoothness in the x variable can also be obtained:

Theorem 2. *Let T be a pseudodifferential operator whose symbol satisfies the inequalities*

$$\sup_x \|\partial_{\xi_j}^{\alpha_j} \partial_{\eta_k}^{\beta_k} \sigma(x,\cdot,\cdot)\|_{L^2(\mathbf{R}^n\times\mathbf{R}^n)} \leq C \tag{5}$$

for all $j,k = 1,\ldots,n$, and $\alpha_j, \beta_k = 0$ or 1. Then, T can be extended as a bounded operator from $L^2(\mathbf{R}^n) \times L^2(\mathbf{R}^n)$ into $L^1(\mathbf{R}^n)$.

Date: March 1, 2004.

A different approach to this problem is to ask the question of continuity for the bilinear Calderón-Vaillancourt class (1) in the context of *better localized spaces*, such as modulation spaces $\mathcal{M}^{p,q}$. In joint work with C. Heil, K. Gröchenig, and K. Okoudjou, using certain time-frequency analysis techniques, we proved

Theorem 3. *If T is a pseudodifferential operator whose symbol satisfies (1), then T can be extended as a bounded operator from $\mathcal{M}^{p_1,q_1}\times\mathcal{M}^{p_2,q_2}$ into $\mathcal{M}^{p_0,q_0}$, $1/p_1+1/p_2 = 1/p_0, 1/q_1+1/q_2 = 1+1/q_0$.*

This theorem is a consequence of a stronger result which also applies to non-smooth symbols. In particular, Theorem 3 shows by how much $L^2\times L^2$ fails to be mapped into L^1 by the bilinear Calderón-Vaillancourt class: $T : L^2\times L^2 \to M^{1,\infty} \supset L^1$. While the last result has a natural multilinear extension, the proofs of the previous two theorems do not seem to be amenable to such generalizations.

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Strichartz's Conjecture on Hardy-Sobolev Spaces

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Our aim is to prove Strichartz's conjecture regarding a characterization of Hardy-Sobolev spaces that arise as an alternative of L^p Sobolev spaces.

- Let H^p be the Hardy spaces on $\mathbb{R}^n$ and I_α the Riesz potentials. The image spaces of H^p under I_α, denoted as $I_\alpha(H^p)$, are called the homogeneous Hardy-Sobolev spaces. For $f \in I_\alpha(H^p)$, there exists a unique $g \in H^p$ with $f = I_\alpha(g)$ and we define $\|f\|_{I_\alpha(H^p)} = \|g\|_{H^p}$.

- For $m \in \mathbb{N}$ and $y \in \mathbb{R}^n$, let Δ_y^m be the mth difference operator defined as $\Delta_y^m f(x) = \Delta_y\big[\Delta_y^{m-1} f\big](x)$, $\Delta_y f(x) = f(x+y) - f(x)$. Let $Q_y^m f(x) = f(x+y) - \sum_{|\sigma|<m} (\partial^\sigma f)(x)\, y^\sigma/\sigma!$. In connection with characterizing $I_\alpha(H^p)$, Strichartz considered a pair of square functions
$$D_{m,\alpha}(f)(x) = \left(\int_0^\infty \left[\int_{|y|<1} \left|\Delta_{ry}^m f(x)\right| dy\right]^2 \frac{dr}{r^{1+2\alpha}}\right)^{1/2},$$
$$T_{m,\alpha}(f)(x) = \left(\int_0^\infty \left[\int_{|y|<1} \left|Q_{ry}^m f(x)\right| dy\right]^2 \frac{dr}{r^{1+2\alpha}}\right)^{1/2}$$
and proved that if $f \in I_\alpha(H^p)$ for $n/(n+\alpha) < p \le 1$, then $D_{m,\alpha}(f) \in L^p$ with $0 < \alpha < m$ and $T_{m,\alpha}(f) \in L^p$ with $m-1 < \alpha < m$. For the reverse direction, he pointed out

  Conjecture. *If either* $D_{m,\alpha}(f) \in L^p$ *with* $0 < \alpha < m$ *or* $T_{m,\alpha}(f) \in L^p$ *with* $m-1 < \alpha < m$, *then* $f \in I_\alpha(H^p)$ *for* $n/(n+\alpha) < p \le 1$.

  (H^p *Sobolev spaces*, Colloq. Math. 60, pp. 129–139, 1990)

- To prove, we characterize $I_\alpha(H^p)$ by means of certain modification of Lusin or Littlewood-Paley functions in such a way analogous to H^p characterizations. Dominating $D_{m,\alpha}(f)$ or $T_{m,\alpha}(f)$ in terms of our new characterizing functions, we obtain the desired proofs.

Fourier integral operators with caustics

Andrew Comech

Abstract. The caustics of Fourier integral operators are defined as caustics of the corresponding Schwartz kernels (Lagrangian distributions on $X \times Y$). The caustic set $\Sigma(\mathbf{C})$ of the canonical relation is characterized as the set of points where the rank of the projection $\pi : \mathbf{C} \to X \times Y$ is smaller than its maximal value, $\dim(X \times Y) - 1$. We derive the $L^p(Y) \to L^q(X)$ estimates on Fourier integral operators with caustics of corank 1 (such as caustics of type A_{m+1}, $m \in \mathbb{N}$). For the values of p and q outside of certain neighborhood of the line of duality, $q = p'$, the $L^p \to L^q$ estimates are proved to be caustics-insensitive.

The results could be applied to the analysis of the blow-up of the estimates on the half-wave operator just before the geodesic flow forms caustics.

Title: Extrapolation on variable L^p spaces

Speaker: David Cruz-Uribe, SFO   Trinity College

Abstract: The variable L^p spaces are a generalization of the classical Lebesgue spaces. Given an exponent function $p(\cdot) : \mathbb{R}^n \to [1, \infty)$, define $L^{p(\cdot)}$ to be the space of functions such that for some $\lambda > 0$,

$$\int_{\mathbb{R}^n} |f(x)/\lambda|^{p(x)}\, dx < \infty.$$

This is a Banach function space when equipped with the norm

$$\|f\|_{p(\cdot),\Omega} = \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^n} \left(\frac{|f(x)|}{\lambda} \right)^{p(x)} dx \leq 1 \right\}.$$

When $p(x) = p_0$, then $L^{p(\cdot)}$ equals L^{p_0}.

We will discuss sufficient conditions on $p(\cdot)$ for the Hardy-Littlewood maximal operator to be bounded on $L^{p(\cdot)}$, and then show how this yields an extrapolation theorem in the scale of variable L^p spaces. As a consequence we get that a wide variety of classical operators from harmonic analysis—singular integrals, fractional integrals, commutators—are bounded on $L^{p(\cdot)}$ whenever the maximal operator is.

Compactness of Sobolev imbeddings for rearrangement invariant norms

G. P. Curbera (Sevilla) & W. J. Ricker (Eichstätt, Germany)

The classical Rellich–Kondrachov theorem on compactness of the Sobolev imbedding

$$W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega), \quad \Omega \subset \mathbb{R}^n$$

shows that the following phenomena occurs: if we fix the range space to be some space $L^q(\Omega)$ smaller than $L^{n'}(\Omega)$, then the imbedding remains compact as long as we do not reach a domain space which is "too large" (i.e. the endpoint $p = \frac{nq}{n+q}$ is avoided). But, if we fix the range space to be some space $L^q(\Omega)$ larger than $L^{n'}(\Omega)$, then the imbedding is compact for all domain spaces.

We show that a similar phenomena of compactness/noncompactness occurs in the framework of Sobolev imbeddings for rearrangement invariant norms

$$W_0^1 Y(\Omega) \hookrightarrow X(\Omega),$$

where X and Y are rearrangement invariant function spaces on [0,1]. The role of the space $L^{n'}(\Omega)$ is now played by the weak space $L^{n',\infty}(\Omega)$. The techniques are based on the reduction of the rearrangement invariant Sobolev's inequality to boundedness of a kernel operator in one variable, due to Edmunds, Kerman and Pick [*J. Funct. Anal.*, 2000], and on a generalized Poincare's inequality, due to Cianchi and Pick [*Ark. Mat.*, 1998].

The results are contained in [Curbera and Ricker, *Studia Math.*, 2003] and [Curbera and Ricker, preprint, 2004].

Unique continuation for second order elliptic operators: a non-Carleman approach

Laura De Carli, (with S. Hudson)

We prove a unique continuation theorem for the differential inequality

$$|\mathrm{div}(\lambda(x)\nabla u)| \leq |V(x)u(x)|, \quad u \in H^{2,1}(\mathbf{R}^n) \tag{0.1}$$

where $\lambda(x)$ is a Lipschitz continuous matrix which satisfies

$$\langle \lambda(x)v,\ v\rangle \geq 0, \quad v \in \mathbf{R}^n.$$

Under suitable assumptions on $V(x)$, we prove that every $H^{2,1}$ solution of the differential inequality (0.1) which vanishes identically outside of a compact set is $\equiv 0$. We also prove a strong unique continuation theorem for the $H^{2,1}_{loc}$ solutions of the differential inequality

$$|\Delta u(x)| \leq |V(x)u(x)|.$$

which is a special case of (0.1). Our proofs do not use Carleman type inequalities.

THE STATIONARY NAVIER-STOKES SYSTEM IN NONSMOOTH MANIFOLDS

Martin Dindoš and Marius Mitrea

The Navier-Stokes equation is one of the most studied nonlinear equation. It models the flow of an incompressible viscous fluid.

Due to nonlinear nature of these equations, one ingredient in their analysis is the study of stationary linearized systems. Traditionally, the linearized system known as the Stokes system is the primary object of such analysis. In our work we introduce and study whole family of such linearized systems among which the Stokes system is only one of many others.

We find the solution to such linear system by the method of layer potentials. The whole approach works in very general framework of Lipschitz domains on Riemannian manifolds. We are particulary interested in what range of Sobolev-Besov spaces the solution exists. Our results show that the smoothness of the solution depends on the smoothness of the boundary, that is for domains with C^1 boundary or boundary with small Lipschitz constant we get solvability in full range of spaces. In general Lipschitz domain this range becomes restricted.

Next we apply the Schauder fixed point theorem to obtain the existence result for the stationary Navier-Stokes equation. Our result shows existence for arbitrary large data in up to four dimensions and small data in higher dimensions.

We will also present several open problems as well as suggest possible ways to solve them.

Some recent estimates for the Ahlfors-Beurling operator

Oliver Dragičević

We will present certain recent results regarding the Ahlfors-Beurling transform

$$Tf(z) = -\frac{1}{\pi}\,\text{p.v.}\int_{\mathbb{C}} \frac{f(\zeta)}{(z-\zeta)^2}\,dA(\zeta)\,.$$

The first result is its estimate on spaces with Muckenhoupt weights:

Theorem *For any $1 < p < \infty$ there is $C(p) > 0$ such that for all $w \in A_p$,*

$$\|T\|_{B(L^p(w))} \leqslant C(p)\cdot\begin{cases} Q_{w,p}^{\frac{1}{p-1}} & ; \quad 1 < p \leqslant 2 \\ Q_{w,p} & ; \quad p \geqslant 2 \end{cases}$$

and this is sharp.

This gave answer to the question of Astala, Iwaniec and Saksman about regularity of solutions of the Beltrami PDE, more precisely, it implied that the weakly quasiregular maps are quasiregular.

Furthermore, we give recent (and so far best) estimates of T on unweighted L^p spaces. They are aimed at the following conjecture (Iwaniec, 1982):

$$\|T\|_p = p - 1 \quad \text{for } p \geqslant 2\,.$$

If proven, this conjecture would have interesting geometric consequences. It would also have implications regarding the well-known Morrey's quasiconvexity problem.

Our results are:

$$\frac{\|T\|_p}{p-1} \leqslant 2$$

$$\limsup_{p\to\infty} \frac{\|T\|_p}{p-1} \leqslant \sqrt{2}$$

$$\lim_{p\to\infty} \frac{\|T\|_{L^p_{real}}}{p-1} = 1$$

These questions are known to be closely related to the theory of quasiconformal mappings, the regularity problems of certain PDE's and some other areas. In our exposition we will also mention the role that martingales and martingale transforms play in these results. In particular, we used certain sharp estimates for these operators to construct a particular Bellman function, which in turn represented the core of some of our proofs. Finally, we will discuss the possibilities of further applications of martingale transforms in this context.

The quoted results were obtained in different papers by Stefanie Petermichl, Alexander Volberg and the above author.

Aspects of harmonic analysis on Damek-Ricci space

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Abstract: Let N be an H-type group and $S = NA$ be its canonical one-dimensional solvable extension.For $\lambda \in \mathbb{C}$, let ϕ_λ the spherical function on S, that is the eigenfunction of the Laplace-Beltrami operator Δ,normalized by $\phi_\lambda(e) = 1$. For an appropriate function on S, the generalized spectral projection operator: $\mathbb{P}_\lambda f(x) = f * \phi_\lambda(x)$ is an eigenfunction of Δ with the same eigenvalue as ϕ_λ.From the Fourier-Helgason inversion formula on $\mathcal{C}_c^\infty(S)$, we have the synthesis and decomposition formulas for functions in terms of eigenfunctions of Δ on S.

$$f(x) = \frac{c_{m,k}}{4\pi} \int_{-\infty}^{+\infty} \mathbb{P}_\lambda f(x) |c(\lambda)|^{-2} \, d\lambda, \;\; where \; \mathbb{P}_\lambda f(x) = \int_N \mathcal{P}_{-\lambda}(x,n) \widehat{f}(\lambda, n) \, dn$$

with $c_{m,k} = 2^{k-1}\Gamma(\frac{2m+k+1}{2})\frac{1}{\pi^{(\frac{2m+k+1}{2})}}$, where $m + k$=dim$N$.

$c(\lambda)$ is the generalized Harish-chadra function, $\widehat{f}$ the Fourier-Helgason transform of f on S and the kernel $\mathcal{P}_\lambda$ is an appropriate complex power of the poisson kernel on S recognized as having connection with the geometry of S [1],[3].

The key question is: how are the properties of f and $\mathbb{P}_\lambda f$ related?.

we characterize the range of $\mathbb{P}_\lambda$ on $\mathcal{C}_c^\infty(S)$ i.e, The function $\mathbb{P}_\lambda$ can be uniquely characterized by analyticity and growth condition in λ of Paley-Wiener theorem type, also, we discuss eigenfunctions of Δ that are generalized spectral projection of L^2-functions on S, the obtained characterizations generalizes the results due to R.S.Strichartz [4] and W.O.Bray [2] on rank-one symmetric spaces of noncompact type.

References

1)A.Abouelaz and O. El Fourchi,Horocyclic Radon transform on Damek-Ricci space, Bulletin of the polish academy of sciences., Vol 2, N 49 (2001).
2)W.O.Bray,Generalized Spectral projection on symmetric spaces of non compact type:Paley-Wiener theorems,J.Geom.Anal.135, N 1(1996)206-232.
3)E.Damek and F. Ricci,Harmonic analysis on solvable extension of H-type groups, J. Geom. Anal. 2, N 3 (1992) 213-248.
4)R. S.Strichartz,Harmonic analysis as spectral theory of Laplacians, J. Funct. Anal. 87 (1989) 51-148.(+Correction in J. Funct. Anal. 109 (1992) 457-560).

On the Wavelet Transform of Almost Periodic Functions

Félix Galindo [1]

Summary

The wavelet transform tries to express and understand the time-frequency behaviour of functions in terms of traslation and scaling of a single function ψ, called *wavelet*. The wavelet transform is usually studied for the analysis of transient signals, in particular, of signals in L^2. It has been used much less frequently in the theory of persistent signals, such as the almost periodic ones, despite the fact that almost periodic processes play an important role in several branches of mathematics, physics and engineering.

The only necessary condition for the wavelet transform of a bounded function with respect a wavelet ψ to be well defined is that ψ belongs to L^1. J.R. Partington and B. Ünalmış use this remark in [1] to consider the wavelet transform of almost periodic functions. Some properties of the wavelet transform as a norm preserving mapping are given in the form of various Parseval relations, but inversion results are not provided. Moreover, it does not seem appropriate to apply the Fourier transform using L^1-functions for the analysis of almost periodic functions. In this talk we put forward a different approach: the space AP of the almost periodic functions can be identified with the space of continuous functions on a compact group G, the Bohr compactification of $\mathbb{R}$. The Fourier transform on G gives the *Bohr-Fourier transform* of $f \in AP$ defined as

$$\hat{f}(\lambda) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(x) e^{-2\pi i \lambda x}\, dx.$$

All of this suggests to use the following definition:

Definition 1 *Let ψ be a function in AP. If f belongs to AP, we define the AP-wavelet transform of f with respect to ψ, denoted by $Wf(t,w)$, as*

$$Wf(s,t) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(x) \overline{\psi\Big(\frac{x-t}{s}\Big)}\, dx, \quad s \in \mathbb{R}\setminus\{0\}, t \in \mathbb{R}. \tag{1}$$

For every $s,t \in \mathbb{R}$, $s \neq 0$, the function $\psi_{s,t}(x) = \psi\left(\frac{x-t}{s}\right)$ belongs to AP and the limit in the definition above makes sense.

With the norm on AP defined by

$$\|f\| = \Big(\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \|f(x)\|_H^2\, dx\Big)^{1/2}, \tag{2}$$

we provide a Parseval formula:

Theorem 1 *If $f \in AP$, with $\hat{f}(0) = 0$, then*

$$\sum_{s\neq 0} \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |Wf(s,t)|^2\, dt = \|f\|^2 \|\psi\|^2\,. \tag{3}$$

And, by means of the polarization identity, we obtain an inversion formula:

Corollary 1 *If $f \in AP$, with $\hat{f}(0) = 0$, we have that*

$$f = \frac{1}{\|\psi\|^2} \sum_{s\neq 0} \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} Wf(s,t)\psi_{s,t}\, dt \tag{4}$$

in the weak sense.

[1] J.R. Partington and B. Ünalmış, "On the Windowed Fourier Transform and Wavelet Transform of Almost Periodic Functions", *Appl. Comput. Harmon. Anal.*, (1) 10 (2001), 45–60.

[1] Research was partially supported by Junta de Castilla y León Grant VA031/02.

Product Rule and Chain Rule Estimates for Hajlasz Gradients on Doubling Metric Measure Spaces

A. Eduardo Gatto and Carlos Segovia Fernández

Abstract

In this paper we introduced Maximal Functions $N(f,x)$ of A.P. Calderón in the context of doubling metric measure spaces (X,d,μ). It is shown that these maximal functions are equivalent to the Hajlasz gradients. Using these maximal functions we prove L^s-norm estimates for the Product Rule and the Chain Rule for functions on (X,d,μ).

SINGULARITY OF ORBITS IN CLASSICAL LIE ALGEBRAS

SANJIV KUMAR GUPTA

Abstract

We determine the maximum k such that the k-fold sum of some non-trivial, adjoint orbit in the Lie algebra of a classical, compact Lie group has measure zero. The orbits of minimal dimension are seen to be the extreme examples. We show that for this choice of k there is a central, continuous measure μ on the group such that μ^k is singular to L^1. For Lie groups other than type B_n or C_3 this result is sharp.

Summary

Let G be a classical, compact, connected, simple Lie group and $\underline{\underline{g}}$ its Lie algebra. Given H in the torus of $\underline{\underline{g}}$ we let O_H denote the adjoint orbit of H,

$$O_H = \{Ad(g)H : g \in G\} \subseteq \underline{\underline{g}}.$$

Adjoint orbits are submanifolds of proper dimension in the Lie algebra and hence have measure zero. Geometric properties of the algebra ensure that if a suitable number of (non-trivial) orbits are added together the resulting subset of $\underline{\underline{g}}$ has positive measure and even non-empty interior. Recently, K. Hare and myself have shown that the least integer k such that k-fold sum of any non-trivial orbit has positive measure (or non-empty interior) is $rank(G) + 1$ if G is of Lie type A_n and $rank(G)$ otherwise. Moreover, we prove that it is the orbits of minimal dimension which are the extreme examples.

This project was originally motivated by the striking result of Ragozin [**?**] who showed that if μ was any central, continuous measure on G, then $\mu^{\dim G} \in L^1(G)$. (Here the product is convolution.) He obtained this by using algebraic and geometric properties of the groups to prove that the $\dim G$-fold product of any non-trivial conjugacy class in the group has positive measure, and **he asked if $\dim G$ could be replaced by a smaller value, in either problem**. His work implies that any sum of $\dim G$ non-trivial orbits has non-empty interior.

Ragozin's results were improved by K. Hare, D. Wilson and W.L. Yee[HWY] using the representation theory of Lie groups. They showed that $\dim G$ could be replaced by $rank(G)+1$ if G was type A_n, $rank(G)$ if G was type $C_n, n > 3$

or D_n, and $2rank(G)$ if G was type B_n, but it was unknown if these values were sharp.

Our work establishes that on any classical, compact, connected, simple Lie group G there is a central, continuous measure μ with μ^k singular to $L^1(G)$, and a conjugacy class whose k-fold product has measure zero, provided $k < rank(G) + 1$ if G is type A_n and $k < rank(G)$ otherwise. Combined with [**?**], this resolves Ragozin's questions for Lie groups of type A_n, C_n with $n > 3$ and D_n. The measures with the extreme property are the orbital measures supported on the exponential of the orbit of minimum dimension in the algebra. These measures also satisfy the dichotomy that $\mu^k \in L^1(G)$ if and only $\mu^k \in L^2(G)$. It would be interesting to know which orbital measures have this dichotomy.

Type B_n seems to be peculiar. The conjugacy class in the group which is minimal in dimension is not the exponential of the orbit of minimum dimension and actually has much smaller dimension. Because of this property the sharp answers to Ragozin's questions on Lie groups of type B_n are still open.

Some of our work has been published in:

1. Sanjiv Kumar Gupta, Kathryn Hare, "Singularity of Orbits in Compact Lie Groups", Journal of Geometric and Functional Analysis, **13**, 815-844, (2003).

2. Sanjiv Kumar Gupta, Kathryn Hare, "Singularity of Orbits in Compact Lie Groups", Israel Journal of Mathematics, **130**, 93-107, (2002).

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Proposal for Talk at El Escorial 2004

Orlicz Bounds for Operators of Restricted Weak Type

Paul Hagelstein

Let T be a sublinear operator mapping the set of measurable functions supported on the unit circle $\mathbb{T}$ into itself. It will be shown that if T is of restricted weak type $(1,1)$, then T is a bounded operator from simple functions in $L \log L\,(\mathbb{T})$ into weak $L^1\,(\mathbb{T})$. Moreover, it will be shown that if T is a sublinear translation-invariant operator of restricted weak type $(1,1)$, then T is a bounded operator from simple functions in $L \log L\,(\mathbb{T})$ into $L^1\,(\mathbb{T})$ itself. This result will be seen to be sharp in view of the recent construction by Hagelstein and R. L. Jones of a sublinear translation-invariant operator T acting on $L^1\,(\mathbb{T})$ which is of restricted weak type $(1,1)$ and maps $L \log L\,(\mathbb{T})$ boundedly into $L^1\,(\mathbb{T})$, but is not of weak type $(1,1)$.

On modular inequalities in variable L^p spaces

Andrei Lerner

Let $p : \mathbb{R}^n \to [1, \infty)$ be a measurable function. Denote by $L^{p(\cdot)}(\mathbb{R}^n)$ the Banach space of measurable functions f on $\mathbb{R}^n$ such that for some $\lambda > 0$,

$$\int_{\mathbb{R}^n} |f(x)/\lambda|^{p(x)} dx < \infty,$$

with norm

$$\|f\|_{L^{p(\cdot)}} = \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^n} |f(x)/\lambda|^{p(x)} dx \leq 1 \right\}.$$

It has been shown recently in the papers of Diening, Nekvinda, Cruz-Uribe, Fiorenza, and Neugebauer that for a large class of functions p the Hardy-Littlewood maximal operator M is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$, i.e.,

$$\|Mf\|_{L^{p(\cdot)}} \leq c\|f\|_{L^{p(\cdot)}}. \tag{1}$$

A natural question arises about conditions on p implying a stronger modular inequality

$$\int_{\mathbb{R}^n} (Mf(x))^{p(x)} dx \leq c \int_{\mathbb{R}^n} |f(x)|^{p(x)} dx. \tag{2}$$

Our main result shows that actually (2) is much stronger than (1).

Theorem. *Let* $\operatorname{ess\,inf}_{x\in\mathbb{R}^n} p(x) > 1$ *and* $\operatorname{ess\,sup}_{x\in\mathbb{R}^n} p(x) < \infty$. *Then inequality (2) holds for any* $f \in L^{p(\cdot)}(\mathbb{R}^n)$ *if and only if* $p(x) \sim const.$

An analogous result for a class of singular integrals will be also discussed.

MULTIRESOLUTION ANALYSIS ASSOCIATED TO DIFFUSION SEMIGROUPS: CONSTRUCTION AND FAST ALGORITHMS

RONALD R. COIFMAN, MAURO MAGGIONI

Abstract. We introduce a novel multiresolution construction for efficiently computing, compressing and applying large powers of operators that have high powers with low numerical rank. This allows the fast computation of functions of the operator, notably the inverse, in compressed form, and their fast application.

Classes of operators satisfying these conditions include important differential operators, in any dimension, on manifolds, and in non-homogeneous media. In this case our construction can be viewed as a far-reaching generalization of Fast Multipole Methods, achieved through a different point of view, and of the non-standard wavelet represenation used for compressing Calderón-Zygmund operators and pseudo-differential operators.

We show how the dyadic powers of an operator can be used to induce a multiresolution analysis, as in classical Littlewood-Paley and wavelet theory, and we show how to construct, with fast and stable algorithms, scaling function and wavelet bases associated to this multiresolution analysis, and the corresponding downsampling operators, and use them to compress the corresponding powers of the operator. This allows to extend multiscale signal processing to general spaces (notably manifolds and graphs) in a very natural way, with corresponding fast algorithms.

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A CONSTRUCTIVE APPROACH TO CAFFARELLI'S $C^{1,\alpha}$-RESULT FOR SOLUTIONS TO THE MONGE-AMPÈRE EQUATION

DIEGO MALDONADO

Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be a convex function and let $\partial\varphi$ denote its normal mapping

$$\partial\varphi(x) = \{p \in \mathbb{R}^n : \varphi(x) + p \cdot (y - x) \leq \varphi(y), \, \forall y \in \mathbb{R}^n\}.$$

The Monge-Ampère measure μ_φ associated to φ is (well-)defined on any Borel set E by

$$\mu_\varphi(E) = |\partial\varphi(E)|,$$

where $|\cdot|$ stands for Lebesgue measure. Given a Borel measure μ on $\mathbb{R}^n$, we say that φ is a solution (in the Alexandrov sense) to $\det D^2\varphi = \mu$ in $\mathbb{R}^n$ if $\mu_\varphi = \mu$. Now, for $x \in \mathbb{R}^n, p \in \partial\varphi(x)$ and $t > 0$, a *section* of φ centered in x at height t is the open convex set

$$S_\varphi(x, p, t) = \{y \in \mathbb{R}^n : \varphi(y) < \varphi(x) + p \cdot (y - x) + t\}.$$

μ_φ is said to satisfy the (DC)-*doubling property* if there exist constants $M > 0$ and $0 < \delta < 1$ such that for all sections $S_\varphi(x, p, t)$, we have

$$\mu_\varphi(S_\varphi(x, p, t)) \leq M\mu_\varphi(\delta S_\varphi(x, p, t)),$$

where $\delta S_\varphi(x, p, t)$ denotes δ-dilation of $S_\varphi(x, p, t)$ with respect to its center of mass. In [1], L. Caffarelli proved the $C^{1,\alpha}$ regularity of φ when μ_φ verifies (DC). His proof is based on a compactness argument that does not provide an estimate for α or the $C^{1,\alpha}$ norm of φ on compact sets. The task of finding the explicit size of these constants was posed as an open problem in Villani's recent book (see [2], pp. 141). We accomplish that task and write the estimates in terms of δ and M. This is joint work with Liliana Forzani.

References

[1] Caffarelli L., Some regularity properties of solutions of Monge-Ampère equation, *Comm. Pure Appl. Math.* **44** (1991), 965-969.

[2] Villani C., *Topics in Optimal Transportation.* Graduate Studies in Mathematics. Volume 58. AMS, 2003.

SECOND ORDER ELLIPTIC OPERATORS AND WEIGHTED NORM INEQUALITIES

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In joint work with P. Auscher, we study weighted norm inequalities for objects associated to elliptic operators in divergence form, $Lf \equiv -\mathrm{div}\,(A\,\nabla f)$ where $A = A(x)$ is an $n \times n$ matrix with L^∞ complex entries, defined on $\mathbb{R}^n$, and satisfying the ellipticity (or "accretivity") condition

$$\lambda\,|\xi|^2 \le \mathrm{Re}\,A\,\xi\cdot\bar{\xi} \qquad \text{and} \qquad |A\,\xi\cdot\bar{\zeta}| \le \Lambda\,|\xi|\,|\zeta|$$

for ξ, $\zeta \in \mathbb{C}^n$ and for some $0 < \lambda \le \Lambda < \infty$.
Recently, P. Auscher has characterized the L^p estimates for several operators associated to L, namely, its semigroup, the gradient of the semigroup, functional calculus, Riesz transforms and square functions. It has been proved that there are four critical numbers that rule the L^p behavior of such objects. These critical numbers are $p_\pm(L)$ and $q_\pm(L)$ which correspond respectively to the limits of the intervals of exponents $p \in [1,\infty]$ for which the semigroup $\{e^{-t\,L}\}_{t>0}$ and its gradient $\{\sqrt{t}\,\nabla e^{-t\,L}\}_{t>0}$ are L^p bounded. In general, the operators considered do not possess a kernel in any reasonable sense (but distributional) and one can not expect any regularity of them as Hörmander condition. This difficulty is overcome by using the divergence structure of the operators and taking advantage of the nice decay in the sense of L^p of the semigroup or its gradient. The range of exponents for which the semigroup or its gradient have this decay turns out to be governed by the critical numbers introduced above.
The aim of this talk is to present the machinery developed by P. Auscher and some new weighted norm inequalities for the operators in question. We will see that any of them satisfy estimates on $L^p(w)$ for some range of p's and for weights w satisfying both an A_q condition and an appropriate reverse Hölder inequality. The precise exponents will depend on p and also on the critical numbers introduced above.

Vector-valued Littlewood-Paley-Stein theory for semigroups

Teresa Martínez

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We develop a generalized Littlewood-Paley theory for semigroups acting on L^p-spaces of functions with values in uniformly convex or smooth Banach spaces. We characterize, in the vector-valued setting, the validity of the one-sided inequalities concerning the generalized Littlewood-Paley-Stein g-function associated with a subordinated Poisson symmetric diffusion semigroup by the martingale cotype and type properties of the underlying Banach space. We show that in the case of the usual Poisson semigroup and the Poisson semigroup subordinated to the Ornstein-Uhlenbeck semigroup on $\mathbb{R}^n$, this general theory becomes more satisfactory (and easier to be handled) in virtue of the theory of vector-valued Calderón-Zygmund singular integral operators.

Elliptic boundary value problems on non-smooth domains

Svitlana Mayboroda and Marius Mitrea

We announce sharp results for the Green operators for the Laplacian with Dirichlet and Neumann boundary conditions on Besov and Triebel-Lizorkin scales in Lipschitz domains. In particular, we solve an open problem, raised by E. Stein and collaborators in the early '90's, pertaining to the boundedness of these operators on Hardy spaces in Lipschitz domains.

Given an open, connected domain $\Omega \subset \mathbb{R}^n$, let $\mathbf{G}_D$, $\mathbf{G}_N$ be the solution operators for the Poisson equation for the Laplacian in Ω with homogeneous Dirichlet and Neumann boundary conditions, respectively. The aim of this note is to describe the sharp ranges of indices p, q, α for which

$$\mathbf{G}_D : B^{p,q}_{\alpha}(\Omega) \longrightarrow B^{p,q}_{\alpha+2}(\Omega), \quad \mathbf{G}_D : F^{p,q}_{\alpha}(\Omega) \longrightarrow F^{p,q}_{\alpha+2}(\Omega), \tag{0.1}$$

and

$$\mathbf{G}_N : B^{p,q}_{\alpha,0}(\Omega) \longrightarrow B^{p,q}_{\alpha+2}(\Omega), \quad \mathbf{G}_N : F^{p,q}_{\alpha,0}(\Omega) \longrightarrow F^{p,q}_{\alpha+2}(\Omega), \tag{0.2}$$

are well-defined and bounded, in the case when Ω is an arbitrary, bounded Lipschitz domain. Here $B^{p,q}_{\alpha}$, $F^{p,q}_{\alpha}$ stand, respectively, for the classes of Besov and Triebel-Lizorkin spaces (the subscript "zero" indicates that the distributions are supported in $\overline{\Omega}$, otherwise they are defined in Ω by restricting corresponding distributions from $\mathbb{R}^n$).

The theorems we prove pertaining to the operators (0.1) extend the work of D. Jerison and C. Kenig [JFA; 1995] whose methods and results are restricted to the case $p \geq 1$. Similarly, our main result dealing with the operators (0.2) generalizes the work of E. Fabes, O. Mendez and M. Mitrea [JFA; 1998], and D. Zanger [CPDE; 2000], and answers the open problem # 3.2.21 on p. 121 in C. Kenig's CBMS book in the most complete fashion.

When specialized to the Triebel-Lizorkin scale with $q = 2$ and $\alpha = 0$, these results solve a (strengthened form of a) conjecture made by D.-C. Chang, S. Krantz and E. Stein [JFA; 1993] regarding the regularity of the Green potentials on Hardy spaces in Lipschitz domains. The authors write: "*For some applications it would be desirable to find minimal smoothness conditions on $\partial\Omega$ in order for our analysis of the Dirichlet and Neumann problems to remain valid. We do not know whether $C^{1+\varepsilon}$ boundary is sufficient in order to obtain h^p_r estimates for the Dirichlet problem when p is near* 1." Here we are able to derive such estimates when $\partial\Omega$ is Lipschitz. A sample result reads

$$\partial_{x_j}\partial_{x_k}\mathbf{G}_D : h^p(\Omega) \longrightarrow h^p(\Omega), \quad \partial_{x_j}\partial_{x_k}\mathbf{G}_N : h^p_0(\Omega) \longrightarrow h^p(\Omega), \quad 1 \leq j, k \leq n, \tag{0.3}$$

provided that $1 - \varepsilon < p < 1$, where $\varepsilon = \varepsilon(\Omega) > 0$. This is sharp in the class of Lipschitz domains.

Other corollaries of our results include new proofs and various extensions of: *(i)* the $L^p - L^q_1$ estimates for Green potentials due to B. Dahlberg, and *(ii)* the $L^p - L^p_2$ estimates for Green potentials due to V. Adolfsson, D. Jerison, S. Fromm, and B. Dahlberg, T. Wolff and G. Verchota.

El Escorial 2004
7th International Conference on Harmonic Analysis and PDE

On the energy spectrum for weak solutions to the 3D Navier-Stokes

Anna L. Mazzucato, Pennsylvania State University

We consider the decay at high wavenumbers of the energy spectrum for weak solutions of the three-dimensional Navier-Stokes equation. We observe that known regularity criteria imply that solutions are regular if the energy spectrum decays at a sufficiently fast rate. This result is true also for a certain class of solutions with infinite energy by localizing the Navier-Stokes equation. We consider modified Leray backward self-similar solutions, which belong to this class, and show that their energy spectrum decay at the expected rate. Therefore, this rate of decay is consistent with the appearance of a self-similar singularity.

Fractional integrals on nonhomogeneous spaces

Alexander Meskhi

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Let (X, d, μ) be a nonhomogeneous space with a quasimetric d and a non-doubling measure μ. Suppose that $\mu(X) = \infty$. Necessary and sufficient conditions on a measure μ governing the two-weight inequality for the fractional integral

$$I_\alpha f(x) = \int_X \frac{f(y)}{d(x,y)^{1-\alpha}} d\mu(y), \quad 0 < \alpha < 1,$$

are established, where the weights are of power type. This enables us to generalize the well-known classical theorem of E. M. Stein and G. Weiss concerning the two-weight inequality for the Riesz potential

$$T_\gamma f(x) = \int_{R^n} \frac{f(y)}{|x-y|^{n-\gamma}} dy, \quad 0 < \gamma < n,$$

in the case of non-doubling measures.

A characterization of the class of measures μ which guarantees the boundedness of I_α from $L^p(\mu, X)$ to $L^q(\mu, X)$, $1 < p \le q < \infty$, $0 < \alpha < 1/p$ is also given.

The talk will be based on the paper [1] and the monograph [2].

References

[1] V. Kokilashvili and A. Meskhi, On some weight inequalities for fractional integrals on non-homogeneous spaces (to appear).

[2] D. E. Edmunds, V. Kokilashvili and A. Meskhi, Bounded and compact operators, Kluwer Academic Publishers, Dordrecht, Boston, London, 2002.

Bochner-Riesz Means of Well-Bounded and C^1-Scalar Generators

Pedro J. Miana

Let A be a (unbounded) self-adjoint with $\sigma(A) \subset \overline{(a,b)} \subset \mathbb{R}$ in a Hilbert space $\mathcal{H}$. Then there exists a spectral measure $\Omega \mapsto E(\Omega)$, $\mathcal{B}(\overline{(a,b)}) \to \mathcal{B}(\mathcal{H})$ such that

$$A = \int_{\sigma(A)} \lambda dE(\lambda) \qquad (1)$$

We can also consider a L^∞-functional calculus, $L^\infty(\sigma(A)) \ni f \mapsto f(A) \in \mathcal{B}(\mathcal{H})$,

$$f(A)x = \int_{\sigma(A)} f(\lambda) d(E(\lambda))x, \qquad x \in \mathcal{H}.$$

If X is a Banach space, scalar-type operator are defined when (1) holds. However, the Laplacian is scalar on $L^p(\mathbb{R}) \Leftrightarrow p = 2$. Other concepts are introduced: well-bounded operators, C^1-scalar operators.

In this talk, we considerar generators $-A$ of holomorphic C_0-semigroups of angle $\frac{\pi}{2}$, $(e^{-zA})_{\Re z>0}$ on a Banach space X. We show that A is a well-bounded operator if X is a UMD-space and $\|e^{-zA}\| \leq M$ for $\Re z > 0$. We relate the $C_c^{(k)}$-functional calculus defined in [GP] with Bochner-Riesz means defined by $(e^{-zA})_{\Re z>0}$. We characterize well-bounded operators and C^1-scalar operators in terms of Bochner-Riesz means. We give examples to illustrate our results.

References

[GP] J.E. Galé and T. Pytlik, *Functional calculus for infinitesimal generators of holomorphic semigroups*, J. of Func. Anal. **150** (1997), 307-355.

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The support of a HARP-postdoctoral position in the Network "Harmonic Analysis and Related Problems" in the C.A.U., Kiel, (Germany) is gratefully acknowledged.

Transplantation theorem for Jacobi series in weighted Hardy spaces

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For α, $\beta > -1$ and for nonnegative integers n, the *Jacobi polynomial* $P_n^{(\alpha,\beta)}(x)$ is defined by

$$(1-x)^\alpha(1+x)^\beta P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{2^n n!}\left(\frac{d}{dx}\right)^n\left[(1-x)^{n+\alpha}(1+x)^{n+\beta}\right].$$

We define the function $\varphi_n^{(\alpha,\beta)}(\theta)$ $(0<\theta<\pi)$ by

$$\varphi_n^{(\alpha,\beta)}(\theta) = s_n^{(\alpha,\beta)}(\sin(\theta/2))^{\alpha+1/2}(\cos(\theta/2))^{\beta+1/2}(\theta)P_n^{(\alpha,\beta)}(\cos\theta),$$

where the constant $s_n^{(\alpha,\beta)}$ is taken so that $s_n^{(\alpha,\beta)} > 0$ and $\int_0^\pi \varphi_n^{(\alpha,\beta)}(\theta)^2 d\theta = 1$. The functions $\varphi_n^{(\alpha,\beta)}$ form a complete orthonormal sequence in the L^2 space on $(0,\pi)$ with respect to the Lebesgue measure.

Muckenhoupt (Mem. Amer. Math. Soc., Vol. 64, No. 356, 1986) proved a transplantation theorem for the functions $\{\varphi_n^{(\alpha,\beta)}\}$, which asserts that under certain conditions on the parameters the estimate

$$\left\| \sum_{n=\max\{-d,0\}}^{\infty} a_n \varphi_{n+d}^{(\alpha,\beta)} \right\|_{L^p_{a,b}} \leqq c \left\| \sum_{n=0}^{\infty} a_n \varphi_n^{(\gamma,\delta)} \right\|_{L^p_{a,b}}$$

holds for all $\{a_n\} \in l^2$, where $L^p_{a,b}$ is the L^p-space $(1<p<\infty)$ on $(0,\pi)$ with respect to the measure $\theta^a(\pi-\theta)^b d\theta$.

The purpose of this short talk is to give an extension of Muckenhoupt's theorem to the case $0 < p \leqq 1$ by using weighted H^p space on $(0,\pi)$.

If $\alpha = \beta = 1/2$ and $\gamma = \delta = -1/2$, Muckenhoupt's theorem reduces to the weighted L^p estimate for the conjugate function transform $\sum_n a_n \cos n\theta \mapsto \sum_n a_n \sin n\theta$. Thus our theorem gives an assertion on the boundedness of certain singular integral transform in certain weighted H^p spaces on the domain $(0,\pi)$. For the boundedness of singular integral transform in H^p spaces on a bounded domain of $\mathbb{R}^n$, there has already been a work by Chang, D.-C., Krantz, S. G., and Stein, E. M. (J. Funct. Anal. **114**(1993), 286–347) and by Chang, D.-C., Dafni, G., and Stein, E. M. (Trans. Amer. Math. Soc. **351**(1999), 1605–1661). The H^p spaces used in our thorem is different from (roughly speaking, wider than) the H^p spaces of these papers, apart from the introduction of weight.

ON RIESZ TRANSFORMS FOR LAGUERRE EXPANSIONS

ADAM NOWAK

Abstract

We study the Riesz transform $\mathcal{R}^\alpha = (R_1^\alpha, \dots, R_d^\alpha)$ naturally associated with multi-dimensional Laguerre polynomial expansions of type α. As our main result we prove that if $\alpha \in [-1/2, \infty)^d$ then R_j^α, $j = 1, \dots, d$, are bounded operators in L^p with appropriate measure for $1 < p < \infty$. Moreover, the corresponding L^p constants are independent of the dimension and the type multi-index α. As a consequence we obtain boundedness and convergence results for the corresponding conjugate Poisson integrals.

Riesz transforms and conjugate Poisson integrals for the Laguerre semigroup were first studied by Muckenhoupt [Mu]. However, he worked in the one-dimensional setting and methods he used seem to be inapplicable in higher dimensions. Recently a g-function and Riesz transforms associated with the multi-dimensional Laguerre semigroup were studied by Gutiérrez, Incognito and Torrea, [GIT]. The technique of "transference" exploited there allowed to obtain L^p boundedness results only for a discrete set of "half-integer" multi-indices α. Using a different approach we remove this restriction and take into account all intermediate multi-indices α.

Our methods are analytic and based on the Littlewood-Paley-Stein theory contained in the monograph [St]. We construct appropriate square functions that relate a function and its Riesz transform, and then prove relevant L^p inequalities for these square functions. Noteworthy, the same scheme was exploited by Gutiérrez [Gu], who considered Riesz transforms associated with the multi-dimensional Hermite semigroup. Nevertheless, the case of Laguerre semigroup turns out to be substantially more involved.

References

[GIT] C. E. Gutiérrez, A. Incognito and J. L. Torrea, *Riesz transforms, g-functions, and multipliers for the Laguerre semigroup*, Houston J. Math. 27 (2001), 579–592.

[Gu] C. E. Gutiérrez, *On the Riesz transforms for Gaussian measures*, J. Funct. Anal. 120 (1994), 107–134.

[Mu] B. Muckenhoupt, *Conjugate functions for Laguerre expansions*, Trans. Amer. Math. Soc. 147 (1970), 403–418.

[St] E. M. Stein, *Topics in Harmonic Analysis Related to the Littlewood-Paley Theory*, Annals of Math. Studies, Vol. 63, Princeton Univ. Press, Princeton, NJ, 1970.

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2000 Mathematics Subject Classification: Primary 42C10; Secondary 42B25;

Key words and phrases: Laguerre expansions, g-functions, Riesz transforms, conjugate Poisson integrals.

A one-sided dyadic maximal function in dimension n

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In [7] E. Sawyer introduced the class of weights A_p^+, which characterizes the pair of weights (w, v) such that the one-sided Hardy-Littlewood maximal operator M^+ apply $L^p(v)$ into weak-$L^p(w)$. These classes of weights and their associated theory have been the subject of much study. Different proofs of Sawyer´s results and more general extensions are given in the papers of F. J. Martín-Reyes, Ortega-Salvador and A. de la Torre [4], [2] and [3]. In [1] H. Aimar, L. Forzani and F. J. Martín-Reyes proved the existence of one-sided singular integral operators, and L. de Rosa and C. Segovia have developed great part of the theory of one-sided weighted Hardy spaces (see [5] and [6]). However, as far as the author knows, the theory of one-sided weights has been developed **only** in $\mathbb{R}$.

The results in this note try to give a first step to overcome that limitation. We will define a dyadic one-sided maximal function in $\mathbb{R}^n$ and we characterize the pair of weights (w, v) such that the operator associated with this maximal function applies $L^p(v)$ into weak-$L^p(w)$. Concretely:

if $I = [a, b)$ is a bounded interval we denote $I^+ = [b, 2b-a)$ and $I^- = [2a-b, a)$. If $Q = I_1 \times I_2 \times ... \times I_n$ is a cube in $\mathbb{R}^n$, we denote $Q^+ = I_1^+ \times I_2^+ \times ... \times I_n^+$, and $Q^- = I_1^- \times I_2^- \times ... \times I_n^-$. Given $f \in L^1_{loc}(\mathbb{R}^n)$, we define the dyadic one-sided maximal function $M^{+,d}f(x)$ as

$$M^{+,d}f(x) = \sup_{Q\ dyadic: x\in Q} \frac{1}{|Q|}\int_{Q^+} |f|,$$

In similar way $M^{-,d}f(x) = \sup_{Q\ dyadic: x\in Q} \frac{1}{|Q|}\int_{Q^-} |f|$.

We prove that the operator $M^{+,d}$ is of weak type (p, p) with respect to the pair (w, v) if and only if, there exists a constant C such that for every dyadic cube Q,

$$\int_Q w \left(\int_{Q^+} v^{-\frac{1}{p-1}}\right)^{p-1} \leq C\,|Q|^p\,, \text{ if } 1 < p < \infty,$$

and $M^{-,d}w(x) \leq Cv(x)$ a.e. if $p = 1$.

References

[1] H. Aimar, L. Forzani and F. Martín-Reyes, *On weighted inequalities for singular integrals.* Proc. Amer. Math. Soc. 125, (1995), 2057-2064.

[2] F. J. Martín-Reyes, *New proofs of weighted inequalities for the one-sided Hardy-Littlewood maximal functions*, Proc. Amer. Math. Soc. 117 (1993), 691-698.

[3] F. J. Martín-Reyes and A. de la Torre, *Two weight norm inequalities for fractional one-sided maximal operators*, Proc. Amer. Math. Soc. 117 (1992), 691-698.

[4] F. J. Martín-Reyes, P. Ortega-Salvador, and A. de la Torre, *Weighted inequalities for one-sided maximal functions. Trans. Amer. Math. Soc. 319* (1990), 517-534.

[5] L. de Rosa and C. Segovia, *Weighted H^p spaces for one sided maximal functions,* Contemporary Math. 189, (1995) 161-183.

[6] L. de Rosa and C. Segovia, *Dual spaces for one-sided weighted Hardy spaces*, Rev. Unión Mat. Arg. (1997).

[7] E. Sawyer, *Weighted inequalities for the one-sided Hardy-Littlewood maximal functions*, Trans. Amer. Math. Soc. 297 (1986), 53-61.

A GEOMETRIC CHARACTERIZATION OF INTERPOLATION IN $\widehat{\mathcal{E}'}(\mathbb{R})$

JOAQUIM ORTEGA-CERDÀ

This is a joint work with X. Massaneda and M. Ounaies.

Let A_p denote the algebra of entire functions f in the plane, such that

$$\log|f(z)| \leq A + Bp(z)$$

for some $A, B > 0$. If

$$p(z) = |\operatorname{Im} z| + \log(1 + |z|^2)$$

one gets the space of Fourier transforms of compactly supported distributions in $\mathbb{R}$.

The main result is the geometric description of all interpolating sequences for this space. There is an analytic characterization of the interpolating sequences due to Berenstein and Li from 1995 for more general p. For the weight p in question a necessary geometric condition was found by Ehrenpreis and Malliavin in 1974. It turns out that this condition is also sufficient provided that the sequence is confined to a certain set around the real axis. We obtain a complete characterization by introducing, in addition to the Ehrenpreis-Malliavin condition a condition of Carleson type at the real axis.

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NON-COMMUTATIVE KHINTCHINE TYPE INEQUALITIES ASSOCIATED WITH FREE GROUPS

JAVIER PARCET

ABSTRACT. Let $\mathbf{F}_n$ be the free group with n generators $g_1, g_2, \ldots, g_n$. Let λ stand for the left regular representation of $\mathbf{F}_n$ and let τ be the standard trace associated to λ. Then, given $2 \le p \le \infty$ and a family $a_i, a_2, \ldots, a_n$ in the Schatten class S_p, the non-commutative Khintchine inequalities for free generators can be stated as follows

$$\Big\| \sum_{k=1}^n a_k \otimes \lambda(g_k) \Big\|_{L_p(\mathrm{tr}\otimes\tau)} \simeq \max \left\{ \Big\| \Big(\sum_{k=1}^n a_k^* a_k \Big)^{\frac{1}{2}} \Big\|_{S_p}, \Big\| \Big(\sum_{k=1}^n a_k a_k^* \Big)^{\frac{1}{2}} \Big\|_{S_p} \right\}.$$

The non-commutative Khintchine inequalities for $1 \le p \le 2$ are dual to these ones. Given a positive integer d, let $\mathcal{W}_p(n,d)$ be the closure in $L_p(\tau)$ of the subspace generated by the family of operators $\lambda(g_{i_1} g_{i_2} \cdots g_{i_d})$. A generic element of $S_p \otimes \mathcal{W}_p(n,d)$ is an operator-valued homogeneous polynomial

$$\mathcal{S}_d(\mathcal{A}) = \sum_{i_1, i_2, \ldots, i_d = 1}^n a_{i_1 i_2 \cdots i_d} \otimes \lambda(g_{i_1} g_{i_2} \cdots g_{i_d}).$$

We are interested in describing the operator space structure of $\mathcal{W}_p(n,d)$. To that aim, we characterize the norm of $\mathcal{S}_d(\mathcal{A})$ in $L_p(\mathrm{tr}\otimes\tau)$. Our characterization generalizes the non-commutative Khintchine inequalities, which correspond to the particular case $d = 1$. Briefly, this characterization can be stated as follows

$$\big\| \mathcal{S}_d(\mathcal{A}) \big\|_{L_p(\mathrm{tr}\otimes\tau)} \simeq \max_{0 \le k \le d} \left\{ \Big\| \Big(a_{(i_1 \cdots i_k),(i_{k+1} \cdots i_d)} \Big) \Big\|_{S_p(S_p)} \right\},$$

where $(i_1 \cdots i_k)$ (resp. $(i_{k+1} \cdots i_d)$) is the row (resp. column) index of a matrix with entries in S_p. The constants involved in this isomorphism do not depend on n nor p. If time permits, we shall also explain the connection of these polynomials with the notion of multi-indexed p-orthogonality. Joint work with GILLES PISIER.

Polysplines – A PDE approach to Multivariate Spline Analysis

Hermann Render

Roughly speaking, polysplines of order p are functions on the euclidean space $\mathbb{R}^n$ which are piecewise solutions of the equation $\Delta^p u(x) = 0$ (where Δ^p is the p-th iterate of the Laplace operator Δ) and which obey certain matching conditions on given knot surfaces of codimension 1; for a precise definition and applications of polysplines see the monograph "Multivariate Polysplines. Applications to Numerical and Wavelet Analysis" (Academic Press 2001) by O. Kounchev.

In this talk we give a survey about interpolation results for cardinal polysplines $u : \mathbb{R}^n \to \mathbb{C}$ on strips: by definition, u is then $2p-2$ times continuously differentiable, and $\Delta^p u(x) = 0$ for all x in the open strips $(j, j+1) \times \mathbb{R}^{n-1}$ for each $j \in \mathbb{Z}$. A major result is the fact that interpolation with polyharmonic splines (introduced by W. Madych and S. Nelson in 1990 and a special case of interpolation with radial basis functions) on lattices of the form $\mathbb{Z} \times a\mathbb{Z}^{n-1}$ for a real number $a > 0$ leads in the limit $a \to 0$ in a natural way to polysplines on strips.

The talk is based on joint work with O. Kounchev.

EXTREMAL PROBLEMS FOR POSITIVE DEFINITE FUNCTIONS WITH GIVEN SUPPORT

MIHAIL N. KOLOUNTZAKIS AND SZILÁRD GY. RÉVÉSZ*

We start with the following problem, originating from a question of Paul Turán. Suppose Ω is a convex body in Euclidean space $\mathbb{R}^d$ which is symmetric with respect to the origin. Of all positive definite functions supported in Ω, and with value 1 at the origin, which one has the largest integral? It is probably the case that the extremal function is the indicator of the half-body convolved with itself and properly scaled, but this has been proved only for a few very particular domains so far. We add to this class of known *Turán domains* the class of all *spectral* convex domains. These are all convex domains which have an orthogonal basis of exponentials $e_\lambda(x) = \exp(2\pi i\langle\lambda, x\rangle)$, $\lambda \in \mathbb{R}^d$. As a corollary we obtain that all convex domains which tile space by translation are Turán domains.

From the above problem of Turán, Arestov, Berdysheva and Berens arrived to pose the analogous pointwise extremal problem for intervals in $\mathbb{R}$. That is, under the same conditions and normalizations, and for any particular point $z \in \Omega$, the supremum of possible function values at z is to be found. However, it turns out that the problem for the real line has already been solved by Boas and Kac, who gave several proofs and also mentioned possible extensions to $\mathbb{R}^d$ and non-convex domains as well.

We present another approach to the problem, giving the solution in $\mathbb{R}^d$ and for several cases in $\mathbb{T}^d$. In fact, we elaborate on the fact that the problem is essentially one-dimensional, and investigate non-convex open domains as well. We show that the extremal problems are equivalent to more familiar ones over trigonometric polynomials, and thus find the extremal values for a few cases. An analysis of the relation of the problem for the space $\mathbb{R}^d$ to that for the torus $\mathbb{T}^d$ is given, showing that the former case is just the limiting case of the latter. Thus the hiearachy of difficulty is established, so that trigonometric polynomial extremal problems gain recognition again.

Finally we study the question: "Given an open set Ω, symmetric about 0, and a continuous, integrable, positive definite function f, supported in Ω and with $f(0) = 1$, how large can $\int f$ be?" in arbitrary locally compact abelian groups and for more general domains. We exhibit upper bounds for $\int f$ assuming geometric properties of Ω of two types: (a) packing properties of Ω and (b) spectral properties of Ω. Several examples and applications of the main theorems are shown. Also, we investigate the question of estimating $\int_\Omega f$ over possibly dispersed sets solely in dependence of the given measure $m := |\Omega|$ of Ω. In this respect we show that in $\mathbb{R}$ and $\mathbb{Z}$ the integral is maximal for intervals.

Besov spaces with matrix weights. Overview.

Svetlana Roudenko

April 13, 2004

Abstract:

We develop some aspects of Littlewood-Paley function space theory in the matrix weight setting. More precisely, we define matrix-weighted continuous and sequence Besov spaces, i.e. $\dot{B}_p^q(W)$ and $\dot{b}_p^q(W)$, and establish the norm equivalence between them if the weight W is in a matrix A_p class, $p > 1$. In certain cases (for example, in scalar case) the equivalence holds if the weight is only doubling. However, we show an example that the doubling condition is not in general sufficient to guarantee the above norm equivalence in matrix case. The dual spaces of both continuous and discrete matrix-weighted Besov spaces are identified; in case when W is not necessarily A_p, the duals are stated in terms of reducing operators. Furthermore, we extend the studies to $p < 1$, introduce the A_p type condition for $p < 1$ and show that the norm equivalence holds in this range of p. Finally, we obtain the boundedness of Calderón-Zygmund operators on matrix-weighted Besov spaces, in particular, the Hilbert and Riesz transforms.

THE SUMMARY OF DMITRY RYABOGIN'S TALK

My talk addresses the Fourier transform as a link between volumes of sections and projections, and other characteristics of the bodies, such as the Minkowski functional or the Gaussian curvature. It has been noticed long ago that many results on sections and projections are dual to each other, though methods used in the proofs are quite different and don't use the duality of underlying structures directly. In the paper [KRZ], we attempted to start a unified approach connecting sections and projections, which may eventually explain these mysterious connections. The idea is to use the recently developed A. Koldobsky's Fourier analytic approach to sections of convex bodies (a short description of this approach can be found in [K]) as a prototype of a new approach to projections. The crucial role in the Fourier approach to sections belongs to certain formulas connecting the volume of sections with the Fourier transform of powers of the Minkowski functional. An analog of this formula for the case of projections was found in [KRZ] and connects the volume of projections to the Fourier transform of the curvature function. I will present this formula and show that it has led to Fourier analytic proofs of several results on projections, including the Fourier analytic solution to Shephard's problem. This problem asks whether bodies with smaller hyperplane projections necessarily have smaller volume. The problem was solved independently by Petty and Schneider, and the answer is affirmative in the dimension two and negative in the dimensions three and higher. The transition in the Busemann-Petty problem (asking whether bodies with smaller hyperplane central sections necessarily have smaller volume) occurs between the dimensions four and five. I will show briefly that the transition in both Shephard and Busemann-Petty problems has the same explanation based on similar Fourier analytic characterizations of intersection and projection bodies. I will also talk about a reconstruction of convex bodies from the volumes of their projections [RZ], and the Fourier analytic characterization of projection bodies in terms of sections of the polar body.

References

[K] A. Koldobsky, *Sections of star bodies and the Fourier transform*, Proceedings of the AMS-IMS-SIAM Summer Research Conference in Harmonic Analysis, Mt Holyoke, 2001, Contemp. Math., to appear (see www.math.missouri.edu/~koldobsk/publications).

[KRZ] A. Koldobsky, D. Ryabogin and A. Zvavitch, *Projections of convex bodies and the Fourier transform*, to appear in Israel J. Math.

[RZ] D. Ryabogin and A. Zvavitch, *Reconstruction of convex bodies of revolution from the areas of their Shadows*, accepted to Archiv der Math.

OTHER RIESZ TRANSFORMS IN THE CONTEXT OF GAUSSIAN HARMONIC ANALYSIS

ROBERTO SCOTTO

In Gaussian Harmonic Analysis of $\mathbb{R}^n$ the differential operator to be used is $L = \frac{1}{2}\Delta - x \cdot \nabla$, called the Ornstein-Uhlenbeck differential operator which is selfadjoint with respect to the Gaussian measure $d\gamma = e^{-|x|^2}dx$. A factorization of this operator is as follows

$$L = \delta \cdot \nabla,$$

where $\delta = \frac{1}{2}e^{|x|^2}\nabla(e^{-|x|^2}.) = \frac{1}{2}\nabla - x$ is called the Gaussian derivative.

By trying to mimic what was done in Classic Harmonic Analysis whose differential operator is the Laplacean $\Delta = \nabla \cdot \nabla$, the Higher Order Riesz Transforms associated with ∇ are defined on the Gaussian context as follows

$$\mathcal{R}_\alpha f(x) = D^\alpha I_{|\alpha|/2} f(x)$$

where $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0$, $|\alpha| = \sum_{j=1}^n \alpha_j$, $D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}}$ e $I_{|\alpha|/2} = (-L)^{-|\alpha|/2}$ is the Riesz Potential of order $|\alpha|/2$.

It was proved that these operators are bounded on $L^p(d\gamma)$ for $1 < p < \infty$. Surprisingly these operators need not be weak type $(1,1)$ for all α. It was proved that they are weak type $(1,1)$ if and only if $|\alpha| \leq 2$.

In a joint work with L. Forzani and H. Aimar, we define the Higher Order Riesz Transforms associated with δ as follows

$$\bar{\mathcal{R}}_\alpha f(x) = \delta^\alpha (-\bar{L})^{-|\alpha|/2} f(x),$$

with $\bar{L} = L - I$ and prove that they are bounded on $L^p(d\gamma)$ for $1 < p < \infty$ and are weak type $(1,1)$ for all α.

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A uniform Tomas-Stein restriction theorem for a class of surfaces in $\mathbf{R}^3$

Faruk Abi-Khuzam and Bassam Shayya*

April 15, 2004

Let $n \geq 2$, $b > 0$, $B = \{x \in \mathbf{R}^{n-1} : |x| < b\}$, and $\mathcal{C} = \{\phi \in C^2(B) : \phi \text{ is convex}\}$. For $\phi \in \mathcal{C}$, we let $\Gamma = \Gamma_\phi : B \to \mathbf{R}$ be the hypersurface in $\mathbf{R}^n$ given by $\Gamma(x) = (x, \phi(x))$ and endowed with the measure $d\sigma$ which is the pushforward under Γ of the $(n-1)$-dimensional measure $K_\phi(x)^{1/(n+1)}dx$, where $K_\phi(x) = \det(\text{Hess}\,\phi\,(x))$ is the affine curvature of Γ. We are interested in the following question that was raised by Carbery and Ziesler: Is there a uniform Tomas-Stein restriction theorem for all such Γ? More precisely, is there an estimate of the form

$$\|\widehat{f}\|_{L^2(\sigma)} \leq C\|f\|_{L^{(2n+2)/(n-1)}(\mathbf{R}^n)} \tag{1}$$

for all $(f, \phi) \in \mathcal{S}(\mathbf{R}^n) \times \mathcal{C}$, where C is a constant that is independent of f and ϕ?

Carbery and Ziesler proved that in dimension $n \geq 3$, the condition ϕ convex can not be replaced by K_ϕ nonnegative. When $n = 2$, the two conditions are the same and the estimate (1) is known to be true (this was proved by P. Sjölin.)

Let $\mathcal{C}I([0, b)) = \{\gamma \in C^3([0, b)) : \gamma(0) = \gamma'(0) = 0, \gamma''(t) > 0 \text{ and } \gamma^{(3)}(t) \geq 0 \text{ for } 0 < t < b\}$. In a recent paper, Oberlin proved that in dimension $n = 3$, (1) is true for all $(f, \phi) \in \mathcal{S}(\mathbf{R}^n) \times \mathcal{C}_1$, where

$$\mathcal{C}_1 = \{\phi \in \mathcal{C} : \phi(x) = \gamma(|x|), \gamma \in \mathcal{C}I([0, b)), \sup_{0<t<b} \frac{t\gamma''(t)}{\gamma'(t)} \leq C\}.$$

We prove that in fact (1) is true for all $(f, \phi) \in \mathcal{S}(\mathbf{R}^n) \times \mathcal{C}_2$, where

$$\mathcal{C}_2 = \{\phi \in \mathcal{C} : \phi(x) = \gamma(|x|), \gamma \in \mathcal{C}I([0, b)), \sup_{0<t<b} \frac{\gamma(t)\gamma''(t)}{\gamma'(t)^2} \leq C\}.$$

Clearly, $\mathcal{C}_1 \subset \mathcal{C}_2$, but what is more important to our purposes is that flat surfaces such as $\phi(x) = e^{-1/|x|^m}$, $m = 1, 2, \ldots$, belong to $\mathcal{C}_2$ (for an appropriate b) but not to $\mathcal{C}_1$. In particular, this shows that restriction and decay are not equivalent in $\mathbf{R}^3$.

*Talk to be delivered by the second author.

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On Weak Type Inequalities for Rare Maximal Functions in R^n

by

Alexander Stokolos (DePaul University, Chicago)

Abstract: A weak $L(\log^+ L)^{n-1}$ type inequality

$$|\{Mf(x) > \lambda\}| \leq C \int \frac{|f|}{\lambda} \left(1 + \log^+ \frac{|f|}{\lambda}\right)^{n-1}$$

is a quantitative version of Jessen-Marcinkiewich-Zygmund theorem. It was established by Miguel de Guzmán in 1972. It is easy to see that this is the best possible estimate.

By Rare Maximal Function we understand the maximal function with respect to rectangles whose side length could be any number from a given infinite sparse set of positive real numbers. If this set is dense enough then the Rare Maximal Function is pointwise comparable with the Strong Maximal Function, and thus is of a weak type $L(\log^+ L)^{n-1}$. Generally, a rarefaction of the set of rectangles could improve the estimate for the corresponding maximal function. In the talk we prove that this is not true for the Rare Maximal Functions in R^n for any $n > 1$, i.e. that the rarefaction of the side-length of the rectangles does not improve the properties of the corresponding maximal functions. Thus, we extend some results known for R^2 only.

Abstract

On Marcinkiewicz Integrals and Harmonic Measure

Ignacio de Uriarte y de Tuero

2004

Jones and Makarov gave sharp density estimates for harmonic measure using a modified version of Marcinkiewicz integrals called $\tilde{I}_0$. It was also used by Jones and Smirnov to substantially advance in the Sobolev and quasiconformal removability problems. We generalize $\tilde{I}_0$ to make it account for different densities of sets over which to integrate, in particular giving a different proof than Jones' and Makarov's of its key properties. The version of $\tilde{I}_0$ that we consider is slightly different than theirs, but is easier to manipulate and has the same applications as theirs.

Our proof is more classical than theirs, decomposes the operator into bite-sized chunks, and allows to "read off" immediately the contribution of each Whitney cube. It is more flexible than the previous one and hence, it should have applications to the aforementioned Sobolev and quasiconformal removability problems, since the geometry and combinatorics of these problems and the estimates proved in this thesis are very similar. The techniques used mainly come from harmonic analysis with a certain combinatorics and probability flavor (e.g. stopping times).

Oscillatory Integrals and Curved Kakeya Sets

Laura Wisewell

We discuss an analogue for curved arcs of the Kakeya problem for straight lines, which arises from Hörmander's conjecture about oscillatory integrals in the same way as the straight line case comes from the restriction and Bochner-Riesz problems. We first review the negative results due to Bourgain, then give an idea of some positive results achieved by geometric and combinatorial methods.

Muckenhoupt condition, BMO and Control Theory

Dmitry V. Yakubovich

In this talk, we will discuss some new relations between real analysis, complex analysis and the control theory.

Let X, U be Hilbert spaces and A a generator of a C_0 semigroup on the state space X, whose spectrum is contained in $\operatorname{clos}\Pi_-$, where $\Pi_- = \{z : \operatorname{Re} z < 0\}$. Consider the linear system

$$\dot{x}(t) = Ax(t) + Bu(t)$$

(A and B can be unbounded). We assume it to be *infinite time exactly controllable*, which means that any control function $u(t)$ in $L^2(\mathbb{R}_-, U)$ defines uniquely a continuous solution $x(t)$ on $(-\infty, 0]$ such that $x(-\infty) = 0$, and any state $x(0)$ can be achieved by some control within this class. This property (as well as exact controllability on finite time intervals) is important for the study of the linear-quadratic optimal control problem.

We will formulate an analytic functional model of A (up to similarity) in terms of a "generalized characteristic function" δ of the system (A, B). It is an $\mathcal{L}(U)$-valued bounded analytic function in the left half-plane.

For any continuous nonvanishing function $g : i\mathbb{R} \to \mathbb{C}$, put

$$\overline{w}_1(g) = \limsup_{s\to+\infty} \frac{1}{s^2}\left[\int_0^s - \int_{-s}^0\right] \arg g(iy)\,dy,$$

whenever the limit exists. We call this characteristic *the mean winding number* of g. In fact, other more general characteristics of g can also be considered. If $g(z) = e^{\alpha z}$, $\alpha \in \mathbb{R}$, then $\overline{w}_1(g) = \alpha$.

Assume that the dimension of U is finite: $U = \mathbb{C}^m$, $m \in \mathbb{N}$. Let $G \in L^\infty_{m\times m}(i\mathbb{R})$. The vector Toeplitz operator T_G on the vector Hardy space $H^2(\Pi_-, \mathbb{C}^m)$ with matrix symbol G is defined by the formula

$$T_G f = P_{H^2(\Pi_-,\mathbb{C}^m)}(G \cdot f), \qquad f \in H^2(\Pi_-, \mathbb{C}^m).$$

Theorem 1. *Let $\tau > 0$. Then the system (A, B) is exactly controllable on $[-\tau, 0]$ if and only if the vector Toeplitz operator $T_{e^{-\tau z}\delta}$ is onto.*

Theorem 2. *If G is an $m \times m$ continuous matrix function on $i\mathbb{R}$ such that* $\det G(z) \neq 0$ *on $i\mathbb{R}$ and $T_{e^{-\tau z}G}$ is onto, then* $\tau \geq \overline{w}_1(\det G)/m$..

The proof of Theorem 2 uses the Nevanlinna factorization and the duality between BMO and the real Hardy space $H^1_{\mathbb{R}}$.

Now let (A, B_1) be another control system, where B_1 is defined on U_1 and has the same properties as B. We use the above results to estimate the minimal controllability time of the system (A, B_1) in terms of $\dim U_1$ and the winding number of the generalized characteristic function δ of the initial system (A, B) (δ is supposed to be known). This result applies, for instance, to linear systems with delays.

We will also briefly comment on the relationship between finite time exact controllability, Riesz bases of exponentials on a finite interval and the Muckenhoupt condition (A_2) on a certain entire function, which is defined by the spectrum of A (results by B. Pavlov and others).

Some unsolved problems will be listed.