ESCORIAL2004: Abstracts

June 17, 2004

1 Abstracts of short courses

Aline Bonami: *Uncertainty Principles related to quadratic forms.*

Abstract: In this series of lectures, I will essentially describe new results of Bruno Demange, which simplify and generalize completely the uncertainty principles obtained in the paper "*Hermite functions and uncertainty principles for the Fourier and the windowed Fourier transforms*", by Bonami, Demange and Jaming (Rev. Mat. Iberomericana **19** (2003), 23–55). The presentation itself has been prepared jointly with Bruno Demange. Our starting point will be Hardy's Uncertainty Principle, which asserts that, for a function f which satisfies the two inequalities

$$|f(x)| \leq C(1+|x|)^N e^{-\pi|x|^2} \quad \text{and} \quad |\widehat{f}(\xi)| \leq C(1+|\xi|)^N e^{-\pi|\xi|^2},$$

there exists a polynomial P, which is of degree at most N, such that $f(x) = P(x)e^{-\pi|x|^2}$. We will see that the equivalent statement for distributions is very easy to prove. The same is valid for Beurling's Uncertainty Principle, and for Morgan's Uncertainty Principle. In this last one, the second members of the two inequalities above are replaced respectively by $\exp(-2\pi\frac{a^p}{p}|x|^p)$ and $\exp(-2\pi\frac{b^{p'}}{p'}|\xi|^{p'})$, with p and p' conjugate exponents. One is lead to the following problem: for two non degenerate quadratic forms q, q' in $\mathbb{R}^d$, consider the distributions $f \in S'(\mathbb{R}^d)$ for which

$$e^{\pm\pi q} f \in S'(\mathbb{R}^d) \quad \text{and} \quad e^{\pm\pi q}\widehat{f} \in S'(\mathbb{R}^d).$$

We are interested in Uncertainty Principles, that is, pairs of quadratic forms for which the space of such distributions is small. We will describe all such distributions when $d = 2$ and $q(x, y) = q'(x, y) = 2\pi\langle x, y\rangle$. Roughly speaking, all solutions are obtained from Gaussian functions which are solutions through a small number of simple operations. When we have a complete description of the space of solutions, we say that we have a *Strong Uncertainty Principle*. If we add some integrability conditions on f (that a Gaussian function does not satisfy) and prove that f vanishes, we speak of a *Weak Uncertainty Principle*. We will also give weak uncertainty principles for some pairs of non degenerate quadratic forms. The main tool is classical complex analysis, which appears naturally once one has transformed the conditions through an integral transform which has already been used by Bargmann in representation theory. As a byproduct, we will see that for some pairs (q, q'), the two sets $\{x \in \mathbb{R}^d; |q(x)| < A\}$ and $\{\xi \in \mathbb{R}^d; |q'(\xi)| < A\}$ form a weakly annihilating pair in the following sense. We refer here to the terminology of the book of Havin and Joricke.

Definition. *Let E and Σ be two measurable sets in $\mathbb{R}^d$. We call (E, Σ) a weakly annihilating pair if the function $f \in L^2(\mathbb{R}^d)$ vanishes as soon as f is supported in E and $\widehat{f}$ is supported in Σ. We call (E, Σ) a strongly annihilating pair if there exists some constant C such that, for every function $f \in L^2(\mathbb{R}^d)$,*

$$\|f\|_2^2 \leq C \left(\int_{E^c} |f|^2 dx + \int_{\Sigma^c} |\widehat{f}|^2 dy \right).$$

A strongly annihilating pair is clearly also a weakly annihilating pair, and it has been shown by Shubin, Vakialian and Wolff that the pairs that we are interested in are strongly annihilating pairs for A small enough. We will start with a long introduction on the Uncertainty Principle, containing the theorem of Shubin, Vakialian and Wolff. The Uncertainty Principle has been the object of many studies. But there are still simple questions which remain unanswered. Even if the subject has a long story, a certain number of recent results are fascinating. Most of them are based on complex methods, but there are exceptions, like the theorem of Shubin, Vakialian and Wolff. Apart from the book of Havin and Joricke, we would like to recommend the expository papers of Havin and Folland-Sitaram, as well as fundamental papers of Nazarov. We chose not to report on the considerable work which has been done on Lie groups in order to concentrate on the Euclidean space.

Terence Tao: *Arithmetic progressions in the primes.*

Abstract: In this series of lectures we describe some of the tools used to establish arithmetic progressions of length three and higher in the set of primes. In the first lecture we discuss the Fourier analysis of the Selberg sieve, which can be used to construct a set of "almost primes" which contains the primes. In particular we discuss a restriction theorem and a Hardy-Littlewood majorant property for that sieve.

In the second lecture we revisit Roth's theorem, asserting that sets of integers of positive density have infinitely many progressions of length three. We combine it with the Selberg sieve restriction theory to establish an analogue of Roth's theorem for the primes (i.e., sets of *primes* of positive density have infinitely many progressions of length three), and for other related sets.

In the third lecture we discuss the case of progressions of arbitrary length. For progressions of length 4 it seems possible, at least in principle, to extend the Fourier-analytic arguments of the preceding lectures by the quadratic Fourier analysis technology of Gowers, but for arbitrary length it seems that it is simpler to proceed instead by ergodic theory methods, which we shall discuss here.

Daniel Tataru: *Wave Packets and Nonlinear Wave Equations.*

Abstract: The aim of these lectures is to describe recent work on quasilinear wave equations. On one hand this involves a careful geometric analysis of the Hamilton flow associated to the wave equation. On the other hand, it requires a phase space decomposition of waves as superposition of certain highly localized waves called wave packets.

Xavier Tolsa: *Painlevé's Problem and Analytic Capacity.*

Abstract: In the first part of the course I will talk about the characterization of analytic capacity in terms of Menger curvature and its comparability with γ_+ (which in particular implies that analytic capacity is semiadditive).

The second part of the course will deal about the bilipschitz "invariance" of analytic capacity. Its proof involves a corona type decompositon for non doubling measures. Using this decomposition, one can also prove that if the Cauchy transform is bounded on $L^2(\mu)$, then any sufficiently smooth Calderón-Zygmund operator with odd kernel is also bounded in $L^2(\mu)$.

To follow this course (specially the first part), the following survey papers are available at *www.mat.uab.es/∼xtolsa*: "Analytic capacity and Calderón-Zygmund theory with non doubling measures", "On the semiadditivity of analytic capacity and planar Cantor sets" (joint work with J. Mateu and J. Verdera). The slides of the course will be available as well at the same address.

2 Abstracts of lectures by invited speakers

Kari Astala: *The Dirichlet-to-Neumann map determines an L^∞-conductivity; Calderón's inverse conductivity problem in dimension two.*

Abstract: We show that the Dirichlet-to-Neumann map for the equation $\nabla \cdot \sigma \nabla u = 0$ in a two dimensional bounded domain uniquely determines the bounded measurable conductivity σ. This gives a positive answer to a question of A.P. Calderón from 1980. Earlier the result has been known only for σ sufficiently smooth. At the end we shall discuss the corresponding result for anisotropic conductivities. The work is joint with Lassi Pšivšrinta (Helsinki), and in the anisotropic case with Pšivšrinta and Matti Lassas. The proofs apply geometric complex analysis and, in particular, use at several points the well established connections between the conductivity equation, quasiconformal methods and the elliptic PDE's.

Luis Caffarelli: *Obstacle like problems: geometry and regularity.*

Guy David: *Open questions on the Mumford-Shah functional.*

Abstract: The Mumford-Shah functional was introduced as a tool for image segmentation, but we shall mostly worry about the theoretical question of regularity of edges in the minimal segmentations.

Recall the formula $J(u,K) = H^{n-1}(K) + \int_\Omega |\nabla u|^2 + \int_\Omega |u-g|^2$, where the domain $\Omega \in \mathbb{R}^n$ and the function $g \in L^\infty(\Omega)$ are given. We are interested in the regularity of K when the pair (u,K) minimizes J.

There were not so many recent developments in the subject; the lecture should focus on the description of the corresponding global functional on $\mathbb{R}^n$ (where we only minimize locally the sum of the two first terms), and in particular in dimension $n = 3$ (where simpler questions are not answered yet).

Hans Feichtinger: *Gabor Analysis, the state of the art (and open problems).*

Abstract: The purpose of this talk is to present an overview on some of the basic achievements in Gabor analysis in the last 15 years. While at the beginning of Gabor analysis a lot of emphasis was on the series representation of arbitrary functions on $L^2(\mathbb{R})$ using as building blocks the so-called *Gabor atoms* (obtained from a mother atom g by applying time-frequency shifts from the lattice $\Lambda = a\mathbb{Z} \times b\mathbb{Z}$, mostly with g being the Gauss function), it is by now clear that Gabor theory has its natural setting over LCA groups G. An important aspect of time-frequency analysis is the study of functions and distributions by the behavior of their STFT (= short-time Fourier transform), $V_g(f) = \langle f, \pi(t, \omega g)\rangle$, which a continuous function over the time-frequency plane. If g is some kind of test function on G, then f is allowed to be a distribution on G. Gabor analysis in turn is essentially concerned with questions which arise by sampling the STFT to some time-frequency lattice, i.e. any discrete cocompact subgroup of $G \times G^*$.

Basic facts concern the Janssen representation of the frame operator and the Wexler-Raz condition, which establishes an equivalence between Gabor frames and Gabor Riesz bases. The canonical dual window, which is crucial in reconstructing the function f from the sampled STFT is characterized among all other duals as the element with minimal norm (or also the one closest to g) in the L^2-norm. A similar characterization applies to the canonical tight frame, which in turn is very useful in designing Gabor multipliers (operators arising by pointwise multiplication of Gabor coefficients of a function). Operators likewise can be described using their spreading symbol or their Kohn-Nirenberg symbol.

In all this it is important to take a view-point beyond the pure L^2-theory. As a matter of fact, for general L^2-windows g, even the sampled STFT can be outside of $\ell^2(\Lambda)$ for some functions $f \in L^2$. The class of modulation spaces has turned out the be the appropriate tool for the choice of "good windows" or to describe the continuity properties of Gabor multipliers, and even in order to derive an improved version of the Calderón-Vaillancourt theorem. Among the modulation spaces, the Segal algebra $S_0(G)$ (also called Feichtinger's algebra nowadays) turns out to be one of the most useful ones. It resembles in its properties very much the Schwartz space (e.g. due to its Fourier invariance and the existence of a kernel theorem), while being at the same time a Banach space (even a Banach algebra, pointwise and with respect to convolution). For windows g from $S_0(G)$ one can show that the dual atom depends continuously on the lattice Λ. Moreover it has been established recently using methods from abstract harmonic analysis that a Gabor frame generated by (g, Λ) with $g \in S_0(G)$ has its canonical dual also in the same space, which implies important consequences for the locality of the Gabor frame expansion, such as its robustness against jitter error or even loss of individual samples. Finally one should mention that the theory of localized frames as developed by K. Groechenig and coauthors allows nowadays to work with Gabor families over irregular discrete sets Λ, and to achieve similar qualitative results using Banach algebra methods.

Loukas Grafakos: *A new way of looking at distributional estimates; applications for the bilinear Hilbert transform.*

Abstract: Sharp distributional estimates for the Carleson operator acting on characteristic functions of measurable sets of finite measure were obtained by Hunt. In this lecture we describe a simple method that yields such estimates for general linear and linearizable operators. As an application we discuss how distributional estimates can be obtained for the bilinear Hilbert transform. These estimates indicate that the square root of the bilinear Hilbert transform is exponentially integrable and they also provide endpoint results on products of Lebesgue spaces where one exponent is near 1 or the sum of the reciprocal of the exponents is near 3/2. The proof of these results is based on an improved energy estimate of characteristic functions with respect to sets of tiles from which appropriate exceptional subsets have been removed.

Istvan Gyongy: *Solving stochastic PDEs–Splitting the difference.*

Abstract: We consider linear and semilinear stochastic PDEs of parabolic type. These kind of equations arise in many areas of physics and engineering, they play an important role, for example, in nonlinear filtering theory. First we give a brief introduction to the theory of the Cauchy problem for these equations, and discuss some pleasant and unpleasant features of stochastic PDEs, which their deterministic counterparts do not possess. Then we investigate various methods of solving stochastic PDEs numerically. Our main interest is to study the rate of convergence of numerical schemes. We present some recent results on splitting-up methods, obtained jointly with Nicolai Krylov. We show, in particular, that for deterministic parabolic PDEs the convergence of the numerical solutions, obtained by these methods, can be accelerated to get convergence of any order.

Waldek Hebisch: *Singular integrals on Iwasawa AN groups.*

Nets Katz: *Remarks on the Sums Differences Problem.*

Abstract: We discuss the Sums Differences Problem related to the Kakeya conjecture. We explain why "elementary" methods cannot completely solve the problem. We discuss the relationship of the problem to group theory. We discuss the importance of rationality of the slices. And we raise a number of questions some relevant to the Kakeya problem and some not.

Herbert Koch: *Dispersive estimates for non-selfadjoint operators.*

Abstract: In this talk I will describe ongoing research with D. Tataru. Motivated by questions from unique continuation we study dispersive estimates for operators with complex symbols under curvature conditions on the characteristic set. It is crucial to split the L^2 estimates and the dispersive estimates. The main tool is the construction of a very rough parametrix under curvature conditions, which reduces in applications the question of L^p estimates to proving L^2 estimates. As byproduct we obtain estimates on several spectral projections.

Mihalis Kolountzakis: *The Fuglede Conjecture.*

Abstract: The Fuglede Conjecture asks whether the domains which tile space by translation are the same as those whose L^2 space admits an orthogonal basis of exponentials $\exp(2\pi i\lambda x)$. In this talk we will describe progress on this question during the past 4-5 years, and recent developments including Tao's counterexample in dimension 5. We will emphasize the restriction of the conjecture to convex bodies.

Izabella Łaba: *Distance sets corresponding to non-Euclidean norms.*

Abstract: Let X be the 2-dimensional plane equipped with a non-Euclidean norm in which the unit ball is a convex set K, and let S be a well-distributed subset of X. We address the question of how small the distance set of S in X can be, depending on properties of K. In particular, we prove that there is a well-distributed S whose distance set has bounded density if and only if K is a polygon with finitely many sides, all of which have algebraic slopes in some coordinate system. We also consider the "continuous" version of the problem, i.e. given a planar set E of positive Hausdorff dimension s, how does the dimension of its distance set in X depend on s and on the properties of K?

The results presented in this talk were obtained jointly with Alex Iosevich and with Sergei Konyagin.

Giancarlo Mauceri: *Spectral multipliers for sublaplacians with drift on Lie groups.*

Abstract: I shall present joint work with W. Hebisch and S. Meda on spectral multipliers of right-invariant sublaplacians with drift on an amenable connected Lie group G. The operators we consider are self-adjoint with respect to a measure $\chi\, d\lambda_G$, whose density with respect to the left Haar measure $d\lambda_G$ is a nontrivial positive character of the group G. We show that if $p \neq 2$, then every $L^p(\chi\, d\lambda_G)$ spectral multiplier extends to a bounded holomorphic function on a parabolic region in the complex plane, which depends on p and on the drift. When G is of polynomial growth we show that this necessary condition is nearly sufficient, by proving that bounded holomorphic functions on the appropriate parabolic region which satisfy mild regularity conditions on its boundary are $L^p(\chi\, d\lambda_G)$ multipliers.

Marco Peloso: *On the L^p spectral multiplier for the Hodge Laplacian on the Heisenberg group.*

Abstract: This is a report on joint work with Detlef Müller and Fulvio Ricci. We study the Hodge Laplacian $\Delta_k = dd^* + d^*d$ acting on k-forms on the $(2n+1)$-dimensional Heisenberg group. We prove that if m is a Mihlin-Hörmander multiplier on the positive half-line, with L^2-order of smoothness greater than $n+1/2$, then $m(\Delta_1)$ is L^p-bounded for $1 < p < \infty$. We also discuss the case for higher degree forms and related results on Δ_k as well.

Carlos Pérez: *On some extensions of J.L. Rubio de Francia's extrapolation theorem.*

Abstract: One of the main theorems in modern Harmonic Analysis is the extrapolation theorem of J.L. Rubio de Francia for A_p weights. In this talk we will discuss some recent extensions of this result. Some of the main points will be the following: a) We will start by describing how to extend the extrapolation theorem to the context of A_∞ weights. As an application we will show that this result provides an alternative method to the classical technique of good-λ inequalities. b) We will show that this extension can be used to prove weak-type endpoint inequalities from strong-type inequalities; this phenomenon, in general, does not hold in the classical context. c) Also we will show how to extend the old and new results to the context of rearrangement-invariant Banach function spaces as well as to the context of modular inequalities. d) Finally, we will discuss how to prove some extrapolation theorems within the context of an appropriate class of non A_∞ weights. As an application we will discuss some problems proposed by E. Sawyer.

Peter Sjögren: *Functional calculus for the Ornstein-Uhlenbeck operator.*

Abstract: The Ornstein-Uhlenbeck operator $\mathcal{L}$ is a self-adjoint Laplacian connected with the Gaussian measure in Euclidean space. Its spectrum is the set of natural numbers. Let m be a function defined on this spectrum. Then $m(\mathcal{L})$ is bounded on L^p for the Gaussian measure if m has a holomorphic extension to a cone $|\arg z| < b$ and verifies Mihlin-Hörmander type conditions on the boundary. The sharp value of b is known, and we shall see that one can weaken the conditions by truncating the cone to the right. The proof goes via estimates for the imaginary powers of $\mathcal{L}$. This is joint work with Mauceri and Meda, and the earlier parts also involve García-Cuerva and Torrea.

Gunther Uhlmann: *Boundary rigidity and the Dirichlet-to-Neumann map.*

Abstract: In this lecture we will discuss some recent results on the boundary rigidity problem arising in differential geometry. The problem consists in determining a Riemannian metric on a compact manifold with boundary by knowing the distance function between boundary points. This problem arises also in geophysics in an attempt to determine the structure of the interior of the Earth by measuring the travel times of seismic waves going through the Earth.

We will discuss a connection between the boundary rigidity problem and the inverse problem of determining the Riemannian metric by knowing the Dirichlet-to-Neumann map associated to the Laplace-Beltrami operator.

Jim Wright: *Some remarks on multiparameter analysis and the Newton diagram*

Abstract: In recent work with A. Carbery and S. Wainger, we have examined certain problems with multiparameter structure (e.g., singular Radon transforms along analytic surfaces and analytic homeomorphisms of tori which carry absolutely convergent Fourier series to uniformly convergent ones). In this talk we shall describe the role of the Newton diagram in these problems.