On a problem of Kahane: Higher dimensional variants

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$$A(\mathbb{T}^k) = \left\{ f \in C(\mathbb{T}^k) : \sum_{\underline{n} \in \mathbb{Z}^k} |\hat{f}(\underline{n})| < \infty \right\}$$

Q

$$\Phi : \mathbb{T}^k \to \mathbb{T}^d$$

$$A(\mathbb{T}^d) \to A(\mathbb{T}^k) \quad ?$$

$$f \to f \circ \Phi$$

$$\Phi : \mathbb{T} \to \mathbb{T}$$

$$\Phi\left(e^{2\pi i t}\right) = e^{2\pi i \varphi(t)}, \qquad \varphi(t) = \underbrace{g(t)}_{\text{periodic}} + kt$$

$$g \equiv \text{constant} \iff \Phi \text{ affine}$$

Thm (Beurling-Helson, Leibenzon, Kahane)

$$\begin{aligned} A(\mathbb{T}) &\to A(\mathbb{T}) \\ f &\mapsto f\circ\Phi \end{aligned} \implies \Phi \text{ affine}$$

Kahane

$$\boxed{\begin{array}{ccc} A(\mathbb{T}) & \to & U(\mathbb{T}) \quad ? \\ f & \to & f \circ \Phi \end{array}}$$

$$U(\mathbb{T}) = \left\{ f \in C(\mathbb{T}) : \quad \| S_N f - f \|_{L^\infty} \to 0 \right\}$$

$$\| f \|_U := \sup_N \| S_N f \|_{L^\infty}$$

$$f(t) = \sum c_n e^{2\pi i n t} \rightsquigarrow f \circ \Phi = \sum c_n e^{2\pi i n \varphi} \in \mho(\mathbb{T}) \quad ?$$

$$\| f \circ \Phi \|_{\mho} \leq \sum |c_n| \, \| e^{in\varphi} \|_{\mho}$$

$$\| e^{in\varphi} \|_{\mho} = \mathcal{O}(1) \quad ?$$

$$\|e^{in\varphi}\|_U = \sup_{\substack{x \in \mathbb{T} \\ n, N}} \left| \int_{|t| \lesssim 1} e^{in\varphi(x-t)} \frac{e^{iNt}}{t} \, dt \right| < +\infty \quad (*)$$

Thm 1 (Alpár)

$(*)$ holds for any real-analytic φ

$(*)$ can fail for smooth φ

$$Hf(x,y) = \text{p.v.} \int_{|t| \lesssim 1} f(x-t,\, y-\varphi(t)) \frac{1}{t}\, dt .$$

$$\rightsquigarrow$$

$$m(n,N) = \int_{|t| \lesssim 1} e^{i(n\varphi(t) + Nt)} \frac{1}{t}\, dt \in L^\infty . \quad (**)$$

Thm 2

(**) holds for any real-analytic φ

(**) can fail for smooth φ

Fabes, Nagel, Rivière, Stein, Wainger

$$\int e^{i\psi(t)} \frac{1}{t}\, dt \qquad \psi(t) = n\varphi_x(t) + Nt$$

$$\boxed{\text{Cancellation arises in 2 ways}}$$

From the oscillation of the phase ψ

$$\left| \frac{d^k \psi}{dt^k}(t) \right| \geq \lambda \implies \left| \int e^{i\psi(t)}\, dt \right| \leq C_k \frac{1}{\lambda^{\frac{1}{k}}}$$

$K \geq 2$

Van der Corput

- From the C·Z kernel $\frac{1}{t}$

$$\widehat{p.v\,\frac{1}{t}}\,(\xi) = \int e^{i\xi t}\,\frac{1}{t}\,dt \in L^\infty$$

OR

$$\boxed{\sup_{\xi, a, b} \left| \int_{a \le |t| \le b} e^{i\xi t}\,\frac{1}{t}\,dt \right| < \infty}$$

Proof of Thm 2

$$I = \int_{|t| \lesssim 1} e^{i(n\varphi(t) + N t)} \frac{1}{t}\, dt$$

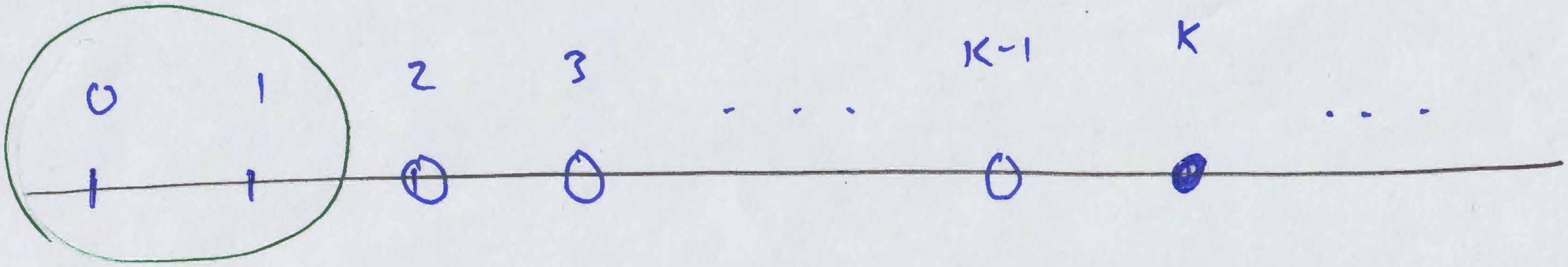
Case 1

$$\varphi^{(k)}(0) = 0 \quad \forall\, k \geq 2 \implies \varphi(t) \equiv a + bt$$

$$|I| = \left| \int_{|t| \lesssim 1} e^{i(nb+N)t} \frac{dt}{t} \right| = O(1)$$

Case 2

$\exists\ k \geq 2$ s.t. $\varphi^{(k)}(0) \neq 0$



$$\varphi(t) = \varphi(0) + \varphi'(0)t \qquad + \qquad \frac{1}{k!}\varphi^{(k)}(0)\, t^k + \dots$$

$$\left| \frac{d^k}{dt^k}\left[n\varphi(t) + Nt \right] \right| \gtrsim n \qquad |t| \leq \varepsilon$$

- $$\int_{\frac{1}{n^{\frac{1}{k}}} \le |t| \le \varepsilon} e^{i(n\varphi(t) + Nt)} \frac{1}{t}\, dt = \mathcal{O}(1)$$ (Van der Corput IBP)

- $$\left| \int_{|t| \le \frac{1}{n^{\frac{1}{k}}}} \left[e^{i(n\varphi(t) + Nt)} - e^{i(n(a+bt) + Nt)} \right] \frac{dt}{t} \right|$$

$$\lesssim n \int_{|t| \le \frac{1}{n^{\frac{1}{k}}}} |t|^{k-1}\, dt = \mathcal{O}(1)$$

Left with

$$\int_{|t| \le \frac{1}{n^{\frac{1}{k}}}} e^{i(nb+N)t} \frac{1}{t} \, dt = O(1)$$

C-Z Kernel cancellation !

Proof of Thm 1

$$I = \int_{|t| \leq 1} e^{i(n\varphi_x(t) + Nt)} \frac{1}{t}\, dt$$

$$\varphi_x(t) = \varphi(x+t)$$

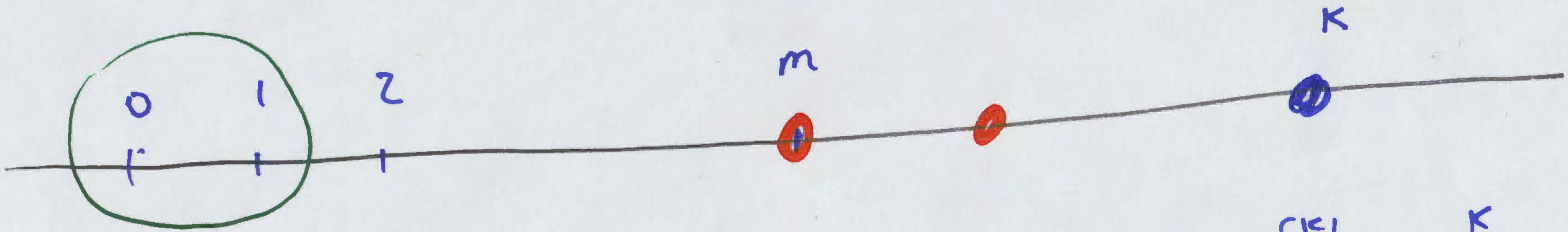
Case 1 $\quad \exists\; x \in \mathbb{T}$ s.t. $\varphi^{(k)}(x) = 0 \quad \forall\, k \geq 2$

$$\Rightarrow \quad \varphi(t) \equiv a + bt$$

✓ C-Z kernel cancellation

Case 2 $\forall x \in \mathbb{T}$, $\exists K = K(x) \geq 2$ s.t. $\varphi^{(K)}(x) \neq 0$

$\exists$ ngh $\mathcal{O}_x$ s.t. $|\varphi^{(K)}(y)| \geq \frac{1}{2} |\varphi^{(K)}(x)| \quad \forall y \in \mathcal{O}_x$



$$\varphi(x+t) = \varphi(x) + \varphi'(x) t + \quad + \frac{1}{K!}\varphi^{(K)}(x) t^K \quad \ldots$$

$$\varphi(y+t) = \varphi(y) + \varphi'(y) t \quad + \quad \frac{1}{m!}\varphi^{(m)}(y) t^m \quad \ldots$$

For $y \in \mathcal{O}_x$, $\quad I = \displaystyle\int_{|t| \le 1} e^{i(n\varphi(y+t)+Nt)} \frac{1}{t}\, dt$

$$\int_{\frac{1}{n^{\frac{1}{k}}} \le |t|} e^{i(n\varphi(y+t)+Nt)} \frac{1}{t}\, dt = \mathcal{O}(1)$$

Van der Corput $(k \geq 2)$

+ IBP

$$\left| \int_{|t| \le \frac{1}{n^{1/k}}} \left[e^{i[n\varphi(y+t)+Nt]} - e^{i\left(n\sum_{l=0}^{k-1} \frac{1}{l!}\varphi^{(l)}(y)\, t^l + Nt\right)} \right] \frac{dt}{t} \right|$$

$$\lesssim n \int_{|t| \le \frac{1}{n^{1/k}}} |t|^{k-1} \, dt = O(1)$$

Left with

$$\int_{|t| \leq \frac{1}{(n)^{\frac{1}{k}}}} e^{i[a_1 t + \cdots + a_{k-1} t^{k-1}]} \frac{1}{t} \, dt = O(1)$$

- Stein-Wanger, Stein, Ricci-Stein

$$Tf(x) = \int_{\mathbb{R}^n} e^{iP(x,y)} K(x,y) f(y) \, dy$$

Corollary of Proof

If $\Phi : \mathbb{T} \to \mathbb{T}^d$ real-analytic

$$A(\mathbb{T}^d) \to U(\mathbb{T})$$

$$f \to f \circ \Phi$$

$$\boxed{\varphi(t) \to \bar{\omega} \cdot \bar{\varphi}(t) = \omega_1 \varphi_1(t) + \dots + \omega_d \varphi_d(t)}$$

$$\Phi : \mathbb{T}^k \to \mathbb{T}^d \quad \text{real-analytic}$$

$$\mathcal{A}(\mathbb{T}^d) \to \mathcal{U}(\mathbb{T}^k) \qquad ??$$

$$f \to f \circ \Phi$$

K=2

$$U(\mathbb{T}^2) = \{ f \in C(\mathbb{T}^2) : \|S_{M,N} f - f\|_{L^\infty(\mathbb{T}^2)} \to 0 \}$$

$$\boxed{\begin{aligned} S_{M,N} f(x,y) &= \sum_{|k| \le M,} \sum_{|l| \le N} \hat{f}(k,l)\, e^{2\pi i k x}\, e^{2\pi i l y} \\ &= \iint_{\mathbb{T}^2} f(x-s, y-t) \frac{\sin(Ms)}{s} \frac{\sin(Nt)}{t}\, ds\, dt \end{aligned}}$$

$k=2, \quad d=1$

C. Fefferman $\quad \exists\, f \in C(\mathbb{T}^2), \quad \overline{\lim_{M,N\to\infty}} \left| S_{M,N} f(x,y) \right| = \infty \quad \forall (x,y) \in \mathbb{T}^2$

$\boxed{\text{Heart}}$ $\quad f_n(x,y) = e^{inxy} \rightsquigarrow \sup_{M,N} \left| S_{M,N} f_n(x,y) \right| \gtrsim \log n$

$\forall (x,y) \in \mathbb{T}^2$

= 2, $d = 1$

$$S_{M,N}(e^{in\varphi})(x,y) = \iint_{|s|,|t|\le 1} e^{in\varphi(x-s,\,y-t)}\,\frac{\sin(Ms)}{s}\,\frac{\sin(Nt)}{t}\,ds\,dt$$

$$\boxed{¿\ \sup_{\substack{M,N,n\\ (x,y)\in\mathbb{T}^2}} \left|S_{M,N}(e^{in\varphi})(x,y)\right| < +\infty\ ?}$$

$$\varphi(s,t) = st \rightsquigarrow e^{in\varphi} = f_n$$

Proposition $\varphi(s,t)$ real-analytic

Suppose $\exists\ (x,y) \in \mathbb{T}^2$ s.t. $\dfrac{\partial^2 \varphi}{\partial s\, \partial t}(x,y) \neq 0$.

Yet $\dfrac{\partial^2 \varphi}{\partial s^2}(x,y) = 0$ $\left(\text{or } \dfrac{\partial^2 \varphi}{\partial t^2}(x,y) = 0 \right)$

Then $$\sup_{M,N} \left| S_{M,N}\left(e^{in\varphi}\right)(x,y) \right| \gtrsim \log n$$

Newton polygon.

$$\boxed{\varphi_{x,y}(s,t) = \varphi(x+s, y+t) = \sum_{k,l\geq 0}\sum \frac{1}{k!}\frac{1}{l!}\frac{\partial^{k+l}\varphi}{\partial s^k \partial t^l}(x,y)\, s^k t^l}$$

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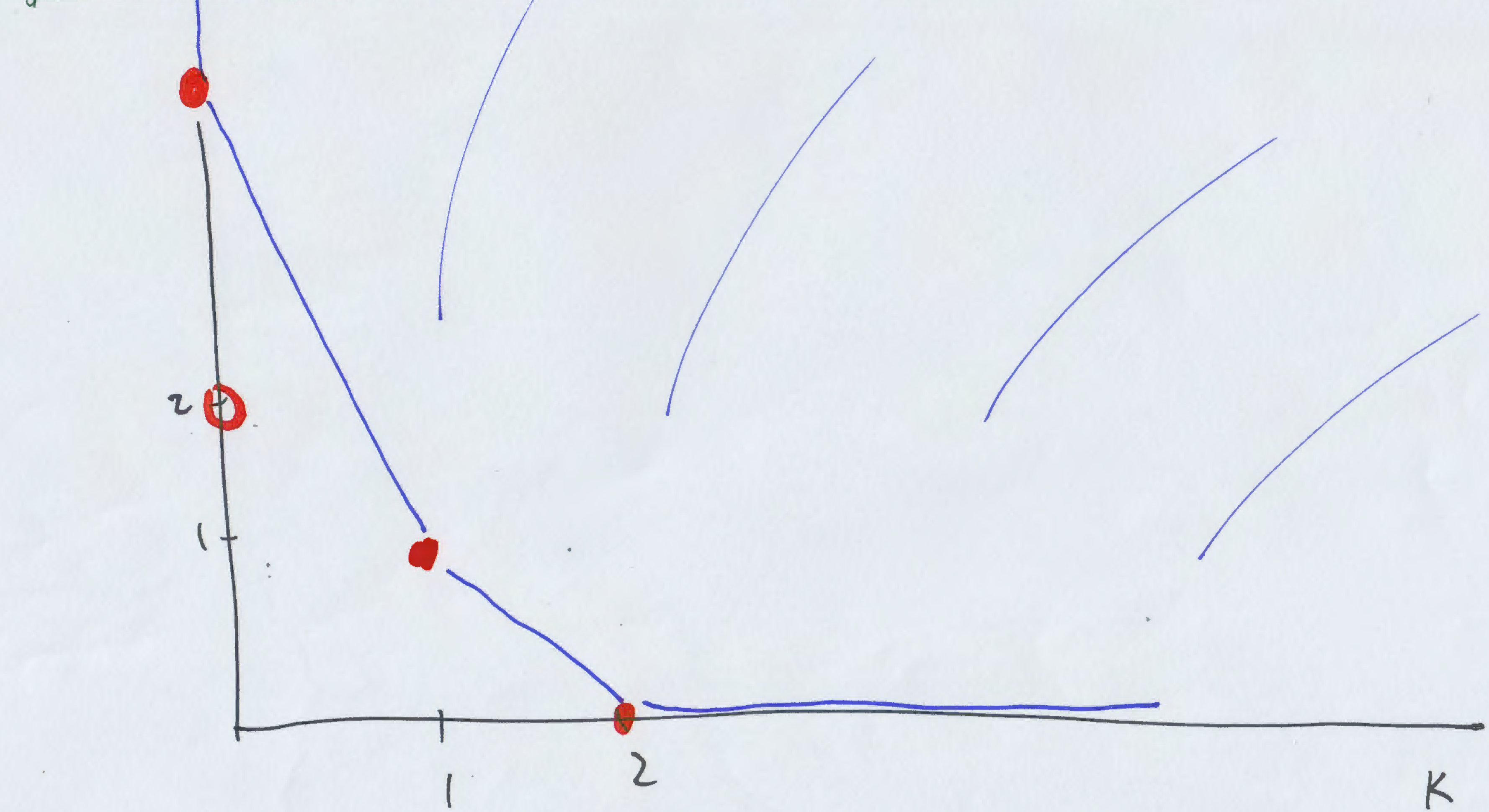

$$Z_{k\ell} = \left\{ (x,y) \in \mathbb{T}^2 : \frac{\partial^{k+\ell}\varphi}{\partial s^k \partial t^\ell}(x,y) = 0 \right\}$$

$$\boxed{\begin{array}{l}\underline{\text{Divisibility condition}} \ (\text{weak form}) \\[1em] \qquad Z_{20} \cup Z_{02} \subseteq Z_{11}\end{array}}$$

Failure $\Rightarrow$ $\displaystyle\sup_{\substack{M,N \\ (x,y)\in\mathbb{T}^2}} \left| S_{M,N}\left(e^{in\varphi}\right)(x,y) \right| \gtrsim \log n$

Examples when D.C. holds

1. $\varphi(s,t) = f(s) + g(t)$

2. $\varphi(s,t) = f(ks + \ell t)$

If φ satisfies D.C., then:

$$\psi = \varphi + \varepsilon K (\varphi_{ss})^3 (\varphi_{tt})^3 \quad \text{satisfies D.C.}$$

Divisibility condition (Strong form)

$\varphi_{ss},\ \varphi_{tt} \mid \varphi_{st}$ as germs of analytic functions

Locally

$$\varphi_{st} = L\, \varphi_{ss}$$

$$\varphi_{st} = K\, \varphi_{tt}$$

Divisibility Proposition

weak form $\iff$ strong form

Proof: Weierstrass Preparation Thm

Osgood, Annals of Math 1917

Thm

$$\overline{\Phi} : \overline{\Pi}^2 \to \overline{\Pi}$$

$$A(\overline{\Pi}) \to A(\overline{\Pi}^2) \quad \Longleftrightarrow \quad \overline{\Phi} \text{ satisfies the Divisibility Condition}$$

$$f \to f \circ \overline{\Phi}$$