

The Riesz transform, quantitative rectifiability, and a two-phase problem for harmonic measure

Xavier Tolsa



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The Riesz and Cauchy transforms

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$$\mathcal{R}_{\mu,\varepsilon} f(x) = \int_{|x-y|>\varepsilon} \frac{x-y}{|x-y|^{n+1}} f(y) d\mu(y).$$

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In \mathbb{C} , the Cauchy transform of $f \in L_{loc}^1(\mu)$ is $\mathcal{C}_\mu f(z) = \lim_{\varepsilon \searrow 0} \mathcal{C}_{\mu,\varepsilon} f(z)$, where

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The existence of principal values is not guaranteed, in general.

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We also denote

$$\mathcal{R}\mu = \mathcal{R}_\mu 1, \quad \mathcal{C}\mu = \mathcal{C}_\mu 1.$$

Rectifiability

We say that E is rectifiable if it is \mathcal{H}^1 -a.e. contained in a countable union of curves of finite length.

E is **n -rectifiable** if it is \mathcal{H}^n -a.e. contained in a countable union of C^1 (or Lipschitz) n -dimensional manifolds.

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Question: Let $E \subset \mathbb{R}^d$ with $\mathcal{H}^n(E) < \infty$, and $\mu = \mathcal{H}_E^n$. If $\mathcal{R}_\mu : L^2(\mu) \rightarrow L^2(\mu)$ is bounded, is then E n -rectifiable?

The Cauchy transform, curvature, and rectifiability

Theorem (David, Léger, 1998)

Let $E \subset \mathbb{C}$ with $\mathcal{H}^1(E) < \infty$, and $\mu = \mathcal{H}_E^1$.

If \mathcal{C}_μ is bounded in $L^2(\mu)$, then E is rectifiable.

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- The AD-regular case when $\mu(B(x, r)) \approx r$ for all $x \in \text{supp } \mu$, $r > 0$, solved previously by Mattila, Melnikov and Verdera.

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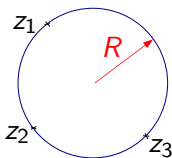
- The AD-regular case when $\mu(B(x, r)) \approx r$ for all $x \in \text{supp } \mu$, $r > 0$, solved previously by Mattila, Melnikov and Verdera.
- It involves ideas from quantitative rectifiability which go back to the Jones' traveling salesman theorem.

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- The proof relies on the relationship between **Menger curvature** and the Cauchy kernel, found by Melnikov.



The curvature of μ is

$$c^2(\mu) := \iiint \frac{1}{R(x, y, z)^2} d\mu(x) d\mu(y) d\mu(z).$$

The David - Léger theorem

Melnikov (1995) found that:

$$\frac{1}{R(z_1, z_2, z_3)^2} = \sum_{s \in S_3} \frac{1}{(z_{s_2} - z_{s_1})(\overline{z_{s_3} - z_{s_1}})}.$$

Melnikov and Verdera (1995): If $\mu(B(z, r)) \leq c r$ for all $z \in \mathbb{C}$, $r > 0$, then

$$\|C_\mu 1\|_{L^2(\mu)}^2 = \frac{1}{6} c^2(\mu) + O(\mu(\mathbb{C})).$$

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The preceding theorem of David and Léger follows from the next one:

Theorem (David, Léger)

If $\mathcal{H}^1(E) < \infty$ and $c^2(\mathcal{H}^1|_E) < \infty$, then E is rectifiable.

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Let $B \subset \mathbb{R}^2$ be a ball such that $C_0^{-1} r(B) \leq \mu(B) \leq C_0 r(B)$.

If

$$c^2(\mu|_B) \leq \varepsilon \mu(B) \quad \text{for some } \varepsilon > 0 \text{ small enough,}$$

then there exists a (possibly rotated) Lipschitz graph Γ with slope $\leq 1/10$ such that

$$\mu(\Gamma \cap B) \geq \frac{99}{100} \mu(B).$$

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- We are interested in finding a version of this result valid in higher dimensions replacing curvature by “the $L^2(\mu)$ norm of $\mathcal{R}\mu$ ”.

Riesz transforms and rectifiability

Theorem (Nazarov, T., Volberg, 2012)

Let $E \subset \mathbb{R}^{n+1}$ with $\mathcal{H}^n(E) < \infty$, and $\mu = \mathcal{H}_E^n$. If $\mathcal{R}_\mu : L^2(\mu) \rightarrow L^2(\mu)$ is bounded, then E n -rectifiable.

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- In the case that μ is AD-regular, i.e.,

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- A previous case solved by Hofmann, Martell, Mayboroda:
For $\mu = \mathcal{H}^n|_{\partial\Omega}$, where Ω is a uniform domain, using harmonic measure and a previous result by Hofmann, Martell, Uriarte-Tuero.

Uniform rectifiability

Let $E \subset \mathbb{R}^d$.

E is **uniformly n -rectifiable** if it is AD-regular and there are $M, \theta > 0$ such that for all $x \in E$, $0 < r \leq \text{diam}(E)$, there exists a Lipschitz map

$$g : \mathbb{R}^n \supset B_n(0, r) \rightarrow \mathbb{R}^d, \quad \|\nabla g\|_\infty \leq M,$$

such that

$$\mathcal{H}^n(E \cap B(x, r) \cap g(B_n(0, r))) \geq \theta r^n.$$

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Uniform n -rectifiability is a quantitative version of n -rectifiability introduced by David and Semmes.

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Recall:

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- (e) For some $0 < \varepsilon \ll 1$, $\int_B |\mathcal{R}\mu(x) - m_{\mu,B}(\mathcal{R}\mu)|^2 d\mu(x) \leq \varepsilon \mu(B)$.

Then there exists some $\tau > 0$ such that if δ, ε are small enough, then there is a uniformly n -rectifiable set $\Gamma \subset \mathbb{R}^{n+1}$ such that

$$\mu(B \cap \Gamma) \geq \tau \mu(B).$$

Remarks

- We have denoted

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- Connection with reflectionless measures.

Key steps of the proof

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3. For the step 1.:
 - We approximate μ by a periodic measure ν .
 - We apply a variational argument, by taking a minimizer of:

$$F(b) = c \|b\|_\infty + \int_B |\mathcal{R}_\nu b|^2 b \, d\nu,$$

with $b \in L^\infty(\nu)$, periodic, such that $\nu(B) = \int_B b \, d\nu$.

Applying a maximum principle we get a contradiction if $\nu(LD)$ is big.

Harmonic measure

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Which is the connection with rectifiability?

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- In the plane if Ω is simply connected and $\mathcal{H}^1(\partial\Omega) < \infty$, then $\mathcal{H}^n \approx \omega^p$. (F.& M. Riesz)
- Many classical results in \mathbb{C} using complex analysis.
- In higher dimension, real analysis techniques. Connection with uniform rectifiability studied recently by Hofmann, Martell, Uriarte-Tuero, Mayboroda, Badger, Bortz, Akman, etc.

Relationship between the Riesz transform and harmonic measure

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Therefore, for $x \in \Omega$:

$$c \nabla_x G(x, p) = K(x - p) - \int K(x - y) d\omega^p(y).$$

That is, $\mathcal{R}\omega^p(x) = K(x - p) - c \nabla_x G(x, p).$

Harmonic measure and rectifiability

By using the Nazarov-T.-Volberg theorem on Riesz transforms and rectifiability, we get a converse to the the F.&M. Riesz theorem:

Theorem

Let $\Omega \subset \mathbb{R}^{n+1}$ be a domain. Suppose that there exists $E \subset \partial\Omega$ with $0 < \mathcal{H}^n(E) < \infty$. If $\omega|_E \approx \mathcal{H}^n|_E$, then E is n -rectifiable.

- Proof by
[Azzam, Mourougolou, T.]
+ [Hofmann, Martell, Mayboroda, T., Volberg].

Application to a two-phase problem

Let $\Omega_1, \Omega_2 \subset \mathbb{R}^{n+1}$ satisfy the CDC, or equivalently,

$$\mathcal{H}_\infty^s((\mathbb{R}^{n+1} \setminus \Omega_i) \cap B(x, r)) \approx r^s \quad \text{for all } x \in \partial\Omega_i \text{ and } 0 < r \leq r_0, \quad (1)$$

for some fixed $s \in (n - 1, n + 1)$, $r_0 > 0$.

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for some fixed $s \in (n-1, n+1)$, $r_0 > 0$.

Theorem (Azzam, Mouroglou, T.)

Let $\Omega_1, \Omega_2 \subseteq \mathbb{R}^{n+1}$ be disjoint connected domains satisfying (1), with harmonic measures ω^1, ω^2 .

Let $E \subseteq \partial\Omega_1 \cap \partial\Omega_2$ be such that $\omega^1 \ll \omega^2 \ll \omega^1$ on E .

Application to a two-phase problem

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Then E contains an n -rectifiable subset F with $\omega^1(E \setminus F) = \omega^2(E \setminus F) = 0$ on which $\omega^1 \ll \omega^2 \ll \mathcal{H}^n \ll \omega^1$.

Remarks

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- Up to now, the theorem was only known when Ω_1, Ω_2 are planar domains by Bishop. Previous case of Jordan domains by Bishop, Carleson, Garnett, Jones.
- A partial result in higher dimensions by Kenig, Preiss and Toro: harmonic measure is concentrated in a subset of dimension n .

Main steps of the proof

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- To show that $\omega^1 \approx \mathcal{H}^n$ on E we apply the Girela-Sarión - T theorem with $\mu = \omega^1|_G$, for suitable subsets $G \subset E$, and we deduce that $\omega^1 \approx \mathcal{H}^n$ on E .

Thank you