Noncommutative de Leeuw theorems

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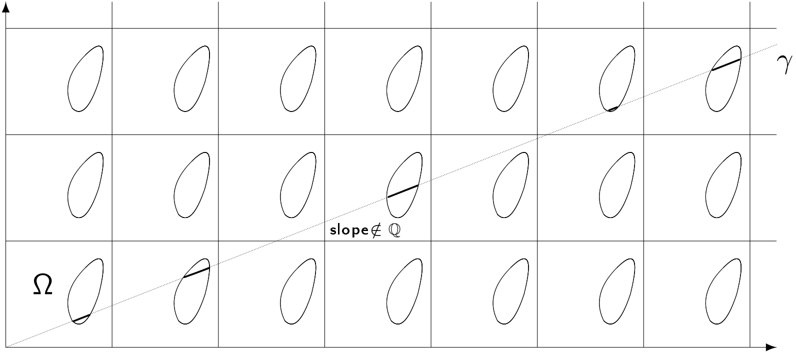
Motivation

Work by M. Junge, T. Mei, J. Parcet on transference of smooth multipliers

To decide if some Idempotent multipliers are bounded on $\mathbb{R}$



De Leeuw's Theorems

General setting : $H \subset \mathbb{R}^d = \widehat{\mathbb{R}^d}$ subgroup, $H = \mathbb{Z}$ and $d = 1$ to simplify.

Question : Relating L_p-multipliers with symbols $m : \mathbb{R}^d \to \mathbb{C}$:

$$\widehat{T_m} : L_p(\widehat{\mathbb{R}^d}) \to L_p(\widehat{\mathbb{R}^d})$$

$$\widehat{T_m f}(\xi) \ = \ m(\xi)\widehat{f}(\xi),$$

$$T_m f \ = \ \int_{\mathbb{R}^n} m(\xi)\widehat{f}(\xi)\lambda(\xi)\, d\xi,$$

to $H \subset \mathbb{R}^d$.

Recall that $\mathbb{R}^d \twoheadrightarrow \widehat{H}$.

De Leeuw's Theorems

The restriction theorem (1965)

$H \subset \mathbb{R}^d$ closed subgroup, $m : \mathbb{R}^d \to \mathbb{C}$ continuous, let $\tilde{m} = m_{|H}$ then

$$\left\| T_{\tilde{m}} : L_p(\widehat{H}) \to L_p(\widehat{H}) \right\| \leqslant \left\| T_m : L_p(\widehat{\mathbb{R}^d}) \to L_p(\widehat{\mathbb{R}^d}) \right\|.$$

Basic case : Transference of multipliers on $L_p(\mathbb{R})$ to $L_p(\mathbb{T})$

m continuous can be weakened :

Riesz Transforms on $L_p(\mathbb{R}) \Rightarrow$ Riesz Transforms on $L_p(\mathbb{T})$

De Leeuw's Theorems

The periodization theorem

$H \subset \mathbb{R}^d$ closed subgroup, $m : \mathbb{R}^d/H \to \mathbb{C}$, let $m_p : \mathbb{R}^d \to \mathbb{C}$ where $m_p = m \circ q$ with $q : \mathbb{R}^d \twoheadrightarrow \mathbb{R}^d/H$ then

$$\left\| T_{m_p} : L_p(\widehat{\mathbb{R}^d}) \to L_p(\widehat{\mathbb{R}^d}) \right\| \leqslant \left\| T_m : L_p(\widehat{\mathbb{R}^d/H}) \to L_p(\widehat{\mathbb{R}^d/H}) \right\|.$$

Basic case : Transference of convolutors on $\ell_p(\mathbb{Z})$ to $L_p(\mathbb{R})$

De Leeuw's Theorems

Let $\mathbb{R}^d_{disc}$ be $\mathbb{R}^d$ with the discrete topology as a LCA group.
$\widehat{\mathbb{R}^d_{disc}} = \mathbb{R}^d_{bohr}$

The compactification theorem

Let $m : \mathbb{R}^d \to \mathbb{C}$ continuous then

$$\Big\| T_m : L_p(\mathbb{R}^d_{bohr}) \to L_p(\mathbb{R}^d_{bohr}) \Big\| = \Big\| T_m : L_p(\widehat{\mathbb{R}^d}) \to L_p(\widehat{\mathbb{R}^d}) \Big\|.$$

Compactification is somehow the strongest result.
It implies the restriction thm as it is clear for discrete G.

All these theorems are about transference.

Ideas behind the proof of compactification

- $\mathbb{R}$ is commutative
- $\mathbb{R}$ is well approximated by discrete subgroups :

$$\mathbb{R} = \overline{\bigcup_{k \geqslant 0} 2^{-k}\mathbb{Z}}.$$

- $\mathbb{R}$ is amenable : existence of Folner sets (intervals)
- Convolution with Gaussians $\gamma(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$, $\gamma_\epsilon = \frac{1}{\epsilon}\gamma\left(\frac{x}{\epsilon}\right)$
  Good approximations on both frequency and space sides.
  Nice semi-group of convolution

Extensions

Saeki 1970

Those 3 theorems hold more generally for LCA groups.

Proof : De Leeuw thms + structure theory of LCA groups.

There are also related works by Igari

There is another approach to compactification :

Extension of multipliers from a closed subgroup to the whole group

Extension of multipliers

Let $F : \mathbb{R} \to \mathbb{R}$, $F = 1_{[-\frac{1}{2},\frac{1}{2}]} * 1_{[-\frac{1}{2},\frac{1}{2}]}$ be a Fejer-type kernel
$F_n^d = F^{\otimes d} : \mathbb{R}^d \to \mathbb{R}$

Jodeit (1969)

Let $m : \mathbb{Z}^d \to \mathbb{C}$ and let $\tilde{m} : \mathbb{R}^d \to \mathbb{C}$ be $m * F^d$, then

$$\Big\| T_{\tilde{m}} : L_p(\mathbb{R}^d) \to L_p(\mathbb{R}^d) \Big\| \leqslant C_d \Big\| T_m : L_p(\mathbb{T}^d) \to L_p(\mathbb{T}^d) \Big\|.$$

Let $m : \mathbb{R}^d \to \mathbb{C}$ and let $\tilde{m}$ be its periodic extension from $[-\pi, \pi]$, then

$$\Big\| T_m : \ell_p(\mathbb{Z}^d) \to \ell_p(\mathbb{Z}^d) \Big\| \leqslant C_d \Big\| T_{\tilde{m}} : L_p(\mathbb{R}^d) \to L_p(\mathbb{R}^d) \Big\|.$$

He also treated restrictions.
This gives the hard way in the compactification theorem.

G LCA, $H \subset G$ be closed.

Figà-Talamanca and Gaudry (1970)

Assume H is discrete, let $m : H \to \mathbb{C}$ and let $\tilde{m} = m * F * F$ for a Fejer-type kernel, then

$$\left\| T_{\tilde{m}} : L_p(\hat{G}) \to L_p(\hat{G}) \right\| \leqslant \left\| T_m : L_p(\widehat{H}) \to L_p(\widehat{H}) \right\|.$$

More satisfactory : Better constant but $F * F$ instead of F.

Cowling (1975)

Assume H is closed, let $m : H \to \mathbb{C}$ and let $\tilde{m} = m * F * F$ for a Fejer-type kernel, then

$$\left\| T_{\tilde{m}} : L_p(\hat{G}) \to L_p(\hat{G}) \right\| \leqslant \left\| T_m : L_p(\widehat{H}) \to L_p(\widehat{H}) \right\|.$$

Use of disintegration and structure theories for LCA groups.
He also looked at periodization.

Noncommutative G ?

$\rightarrow$ Lots of works on convolutors on $L_p(G)$ and $L_p(G/H)$ (Doodley, Gaudry, Derighetti,...)

$\rightarrow$ Lots of works on (vector valued) transference (Coiffman-Weiss,...)

$\rightarrow$ Another direction : Noncommutative analysis

How to define $L_p(\hat{G})$ when G is non abelian ?

For $p = \infty$
$L_\infty(\hat{G}) \subset B(L_2(\hat{G})) \simeq B(L_2(G))$
where $L_\infty(\hat{G})$: convolution operator on $B(L_2(G))$
G l.c. group : left representation $\lambda : G \rightarrow \mathbb{B}(\ell_2 G)$

$$L(G) = \lambda(G)'' = \overline{\mathrm{Span}\ \{\lambda(g)\ ; g \in G\}}^{w}$$

Noncommutative L_p-space

Non commutative analysis

$\mathcal{M} \subset B(\ell_2)$ von Neumann algebra with a normal faithful semifinite trace τ

$$\begin{array}{ccc} (\mathcal{M},\tau) & \leftrightarrow & (L_\infty(\Omega),\mu) \\ \tau(xy)=\tau(yx) & \leftrightarrow & fg=gf \\ \text{normal} & \leftrightarrow & \text{monotone cv thm} \\ \text{faithful} & \leftrightarrow & \text{full support} \\ \text{semifinite} & \leftrightarrow & \sigma\text{-finite} \\ L_0(\mathcal{M},\tau) & \leftrightarrow & L_0(\Omega,\mu) \end{array}$$

via functional calculus on $L_0(\mathcal{M},\tau)$, $|y| = (y^*y)^{\frac{1}{2}}$

$$L_p(\mathcal{M},\tau) = \{x \in L_0(\mathcal{M},\tau) \mid \|x\|_p^p = \tau(|x|^p) < \infty\}$$

Not all vN admit a trace.
If not, equivalent definitions by Connes and Haagerup

Noncommutative G ?

We definie $L_p(\hat{G}) = L_p(L(G))$ for G LC.
Definition of the trace τ via FT (G unimodular)
$x = \int f_g \lambda(g) dg$, $f \in C_c(G)$: $\tau(x) = f_e$.

Notion of multiplier with symbol $\phi : G \to \mathbb{C}$:

$$M_\phi(\int f \lambda(g) dg) = \int \phi_g f_g \lambda(g) dg$$

Restriction, periodization, compactification make sense in this setting.

Noncommutative G?

Pb : What is the best notion of boundedness?

$\rightarrow$ Boundedness of M_ϕ on L_p

$\rightarrow$ Complete Boundedness : boundedness of $M_{\phi\times 1}$ on $G \times G'$ for any G'.

If G and G' are abelian : $M_{\phi\times 1}$ bounded $\Leftrightarrow$ M_ϕ bounded

Arhancet / Dutta-Mohanty-Tewari

For any infinite LCA group G and $1 < p \neq 2 < \infty$, there is an L_p-Fourier multiplier which is not completely bounded.

Extension of a result by Pisier for compact groups using transference.

Cb norm results

For $m : G \to \mathbb{C}$, $T_m : L_p(L(G)) \to L_p(L(G))$ Fourier multiplier
$S_m : S_p(L_2(G)) \to S_p(L_2(G))$ the equivariant Schur multiplier
$\phi(s,t) = m(s^{-1}t)$.

Neuwirth-R (G discrete), Caspers-de la Salle (G LC)

Assume G is amenable and m is bounded

$$\left\| T_m : L_p(L(G)) \to L_p(L(G)) \right\|_{cb} = \left\| S_m : S_p(L_2(G)) \to S_p(L_2(G)) \right\|_{cb}$$

It suffices to look at restriction, periodization, compactification for Schur multipliers.

Using basic results by Haagerup, Lafforgue-de la Salle

An easy compactification theorem

Assume G is amenable, $1 \leqslant p \leqslant \infty$, let $m : G \to \mathbb{C}$ continuous then

$$\Big\| T_m : L_p(L(G_{disc})) \to L_p(L(G_{disc})) \Big\|_{cb} = \Big\| T_m : L_p(L(G)) \to L_p(L(G)) \Big\|_{cb}.$$

An easy restriction theorem

Assume $H \subset G$ with compatible modular functions and H amenable, $1 \leqslant p \leqslant \infty$, let $m : G \to \mathbb{C}$ continuous, $\tilde{m} = m_{|H}$ then

$$\Big\| T_{\tilde{m}} : L_p(L(H)) \to L_p(L(H)) \Big\|_{cb} \leqslant \Big\| T_m : L_p(L(G)) \to L_p(L(G)) \Big\|_{cb}.$$

When $p = 1, \infty$, one can remove H amenable (Bożejko-Fendler).

Similarly there are is an easy periodization theorem.

An easy Jodeit's theorem

Assume G is amenable and $H \subset G$ be a lattice with fd X, $1 \leqslant p \leqslant \infty$, let $m : H \to \mathbb{C}$, put $\tilde{m} = 1_X * m * 1_X$ then

$$\Big\| T_{\tilde{m}} : L_p(L(G)) \to L_p(L(G)) \Big\|_{cb} = \Big\| T_m : L_p(L(H)) \to L_p(L(H)) \Big\|_{cb}.$$

$1_X * 1_X = F \to$ cst 1 : better than Jodeit's result for $\mathbb{Z} \subset \mathbb{R}$ but cb.

Question : What is the right constant in Jodeit's thm (not cb) ?

Drawbacks :

- Amenability of G : hard to get rid of it
- *cb* assumption, one does not recover the classical results

general result for bounded maps $\Rightarrow$ *cb* version

Basic idea's :

- to adapt de Leeuw's approach
- to use other transferences if possible
- to relate $L_p(L(H))$ and $L_p(L(G))$

The basic restriction thm

Assume G is LC and $H \subset G$ amenable discrete with $(\Delta_G)_{|H} = 1$, let $m : G \to \mathbb{C}$ continuous, $\tilde{m} = m_{|H}$

$$\left\| T_{\tilde{m}} : L_p(L(H)) \to L_p(L(H)) \right\| \leqslant \left\| T_m : L_p(L(G)) \to L_p(L(G)) \right\|.$$

Idea of the proof :

To embed $L_p(L(H))$ approximately in $L_p(L(G))$ in a way that intertwines multipliers

$$\begin{array}{ccc} L_p(L(H)) & \xrightarrow{\phi_i} & L_p(L(G)) \\ T_{\tilde{m}} \downarrow & & \downarrow \ T_m \\ L_p(L(H)) & \xrightarrow{\phi_i} & L_p(L(G)) \end{array}$$

Take V a small symmetric neighborhood of $1 \in G$
$y = \frac{1}{\sqrt{\mu(V)}}\lambda(1_V) \in L_2(L(G))$

$$\phi_y^p : L_p(L(H)) \to L_p(L(G)) \ ; \ \lambda(h) \mapsto \lambda(h)u|y|^{2/p}$$

ϕ_y^p is a contraction by interpolation if V is small enough.

One would think of $\lim_{V\to\{e\}} \|\phi_y^p(f)\|_p = \|f\|_p$.

- Obvious for $p = 2$
- Using L_2-duality $\to$ obvious when G is commutative

To get it we need that V is almost invariant by conjugation by H

$$\forall h \in H, \quad \mu(hVh^{-1}\Delta V)/\mu(V) \to 0$$

If this is true G is $[SAIN]_H$

H amenable $\Rightarrow$ G is $[SAIN]_H$

The commutation relation with T_m is a more delicate technical issue.

$\rightarrow$ Obvious for $p = 2$: good space location of y in L_2.
$\rightarrow$ It suffices to do it for ucp T_m using continuity of m.

What is the support of $|y|^t$?

De Leeuw $\rightarrow$ multiplication with γ_ϵ instead of y :
nice convolution semi-group : $\gamma_\epsilon^t = \gamma_{t\epsilon}$
good approximations of identity

One needs a "local control" on approximations of identity for different values of p.

Almost multiplicative maps

Multiplicative domains

Let A be a C^*-algebra, $T : A \to A$ be ucp and $x = x^* \in A$, then

$$T(x^2) = T(x)^2 \quad \Rightarrow \quad T(f(x)) = f(T(x)),\ f \in C(\sigma(x))$$

Using an ultraproduct argument if $\|x\| \leqslant 1$ and $f \in C([-1,1])$

$$\|T(x^2) - T(x)^2\| \leqslant \epsilon \quad \Rightarrow \quad \|T(f(x)) - f(T(x))\| \leqslant \delta$$

If $A = C([0,1])$, this is a strong quantitative Korovkin theorem.

Assume $T : (M,\tau) \to (M,\tau)$ is ucp trace preserving

Then $T : L_p \to L_p$

What can we say if $x \in L_p$?

Almost multiplicativity on L_p

Let $x \in L_p^+$ and $T : M \to M$ ucp τ-preserving then

$$\|T(x) - T(\sqrt{x})^2\|_{2p} \leqslant \frac{1}{2}\|T(x^2) - T(x)^2\|_p.$$

Local approximations of identity

Let $y \in L_2$ with $y = u|y|$ and $T : M \to M$ ucp τ-preserving then

$$\|T(u|y|^\theta) - u|y|^\theta\|_{\frac{2}{\theta}} \leqslant C\|T(y) - y\|_2^{\frac{\theta}{4}}\|y\|_2^{\frac{3\theta}{4}}.$$

This gives the commutation relation !

No easy ultraproduct argument (type III)

More elaborated versions

Recall De Leeuw's idea $\mathbb{R} = \overline{\bigcup_{k \geqslant 0} 2^{-k}\mathbb{Z}}$.

We say that G is ADS if there is are lattices $\Gamma_i \subset G$ with fd X_i shrinking to $\{e\}$.

Examples : LCA, Heisenberg groups, Nilpotent matricial groups.

The restriction thm

Assume G is LC and $H \subset G$ with $(\Delta_G)_{|H} = 1$ and $H \in ADS$, $G \in [SAIN]_H$, let $m : G \to \mathbb{C}$ continuous, $\tilde{m} = m_{|H}$ for $1 \leqslant p \leqslant \infty$:

$$\Big\| T_{\tilde{m}} : L_p(L(H)) \to L_p(L(H)) \Big\| \leqslant \Big\| T_m : L_p(L(G)) \to L_p(L(G)) \Big\|.$$

The compactification theorem

Let $1 \leqslant p \leqslant \infty$, let $m : G \to \mathbb{C}$ continuous,

- If G is ADS

$$\Big\| T_m : L_p(L(G)) \to L_p(L(G)) \Big\| \leqslant \Big\| T_m : L_p(L(G_{disc})) \to L_p(L(G_{disc})) \Big\|,$$

- If G_{disc} is amenable

$$\Big\| T_m : L_p(L(G_{disc})) \to L_p(L(G_{disc})) \Big\| \leqslant \Big\| T_m : L_p(L(G)) \to L_p(L(G)) \Big\|.$$

There is $=$ for LCA, Heisenberg, Nilpotent triangular matricial groups.

One can also get some periodization results.