CIMPA 2017 Research School. Harmonic Analysis, Geometric Measure Theory and Applications

July 31st - August 11, 2017

Objectives

The aim of this school is to focus on those aspects of Harmonic Analysis which recently hav e had a huge impact, in particular in image and signal processing. One characteristic feature is that several technological deadlocks have been solved through the resolution of deep theoretical problems in harmonic analysis and Geometric Measure Theory. It is our purpose to present the new interlaces between Geometric Measure Theory and Harmonic Analysis and how these new undestandings can be applied to solve real life problems. The courses of this CIMPA school will be taught be leaders in these areas and will cover both: theoretical aspects and applications. Our goal is that the courses can be followed by PhD students and PostDocs in mathematics, and in signal and image processing; the challenge being to emulate new interactions between these communities.

Date and Location:

The conference will take place from **July 31st** through **August 11th, 2017** in **Buenos Aires, Argentina**.

IMPORTANT: Registration Deadline: April 9th, 2017.

Conference webpage: **http://mate.dm.uba.ar/~hafg/cimpa-2017**.

Scientific Committee:

- Akram Aldroubi
- Carlos Cabrelli
- Stephane Jaffard
- Ursula Molter

Local Organizing Committee:

- Elona Agora
- Jorge Antezana
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- Ezequiel Rela
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Balian-Low
and
Time Frequency shift invariance

El Escorial 2016

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Def: A SIS (shift invariant space) is a space of functions that is invariant under integer translations.

They are often models for classes of signals and images.

Examples:

I) $PW(\mathbb{R}) := \{ f \in L^2(\mathbb{R}) : \operatorname{supp}(\hat{f}) \subseteq [-\tfrac{1}{2}, \tfrac{1}{2}] \}$

Note: $f \in PW(\mathbb{R}) \Rightarrow t_x f \in PW(\mathbb{R})$

$$t_x f(y) = f(y-x)$$

$PW(\mathbb{R})$ is $\boxed{\text{translation invariant}}$!

Thm [Wiener]: $S \subseteq L^2(\mathbb{R}^d)$ is translation invariant, iff there exists a measurable set $A \subseteq \mathbb{R}$: $S := \{ f \in L^2 : \hat{f}(\omega) = 0 \text{ a.e. } A^c \}$

$$S = \overline{\operatorname{span}}\left\{ t_k \chi_{[0,1]} : k \in \mathbb{Z} \right\}$$

$f \in S$ then $t_x f \in S \iff x \in \mathbb{Z}$.

III) $$S = \overline{\operatorname{span}}\left\{ t_k \chi_{[0,1]},\ t_j\left(\chi_{[0,1/2]} - \chi_{[1/2,1]}\right) : j, k \in \mathbb{Z} \right\}$$

$f \in S$ then $t_x f \in S \iff x \in \frac{1}{2}\mathbb{Z}$.

why? $V_0 \oplus W_0 = V_1$ (MRA)

In ACHRM we gave a characterization for S to have 'extra'-invariance - in particular:

$$M := \{\theta \in \mathbb{R} : t_\theta f \in S \ \forall\, f \in S\}$$

is a subgroup of $\mathbb{R}$ that contains $\mathbb{Z}$.

Proposition *Let S be a SIS. Then either S is translation-invariant, or there exists a maximum positive integer n such that S is $\frac{1}{n}\mathbb{Z}$-invariant.*

$\lambda \in \Lambda \subseteq \mathbb{R} \times \widehat{\mathbb{R}}$, $\varphi \in L^2(\mathbb{R})$, $\Lambda = R\,\mathbb{Z}^2$ (R invertible)

Gabor space $\mathcal{G}(\varphi, \Lambda) = \overline{\operatorname{span}} \{\pi(\lambda)\varphi\}_{\lambda \in \Lambda}$

vs. SIS

Question: Can there exist $\mu \in \mathbb{R} \times \widehat{\mathbb{R}} \setminus \Lambda$ such that $\pi(\mu)\varphi \in \mathcal{G}(\varphi, \Lambda)$?

Look at the Zak-transform:

$$Z\varphi(x, \omega) := \sum_{k \in \mathbb{Z}} \varphi(x+k)\, e^{-2\pi i k\omega}, \quad (x, \omega) \in \mathbb{R} \times \widehat{\mathbb{R}}$$

and note that:

$$Z\varphi(x+m, \omega) = e^{2\pi i m\omega} Z\varphi(x, \omega), \qquad Z\varphi(x, \omega+m) = Z\varphi(x, \omega)$$

Hence: $Z(\pi(k,l)\varphi)(x,\omega) = e^{2\pi i (lx + k\omega)} Z\varphi(x,\omega)$

below

Note that $\varphi \in S_0 \Rightarrow Z\varphi$ continuous.

Assume $\Lambda = \mathbb{Z} \times p\mathbb{Z}$, $p \in \mathbb{N}$, $\mu = (u, \eta)$: $u, \eta \in \mathbb{Q}$

and $\pi(u,\eta)\varphi \in \mathcal{G}(\varphi, \mathbb{Z} \times p\mathbb{Z})$.

We have: $\pi(u,\eta)\varphi \in \mathcal{G}(\varphi, \mathbb{Z} \times p\mathbb{Z}) \Leftrightarrow$

$$Z(\pi(u,\eta)\varphi) \in Z\mathcal{G}(\varphi, \mathbb{Z} \times p\mathbb{Z}) =$$

$$= \overline{\text{span}} \left\{ e^{2\pi i (plx + k\omega)} Z\varphi(x,\omega),\ (k,l) \in \mathbb{Z}^2 \right\}$$

$$S_0 := \left\{ f \in L^2 : Vf(t,\nu) = \int f(x) e^{-(x-t)^2} e^{2\pi i x\nu}\, dx \in L^1_{t,\nu} \right\}$$

Back to SIS:

Aldroubi, Sun, Wang show:

If $\varphi \in L^2$ and $\{t_k \varphi\}_{k \in \mathbb{Z}}$ is a Riesz basis of $S(\varphi)$, if $S(\varphi)$ is translation invariant, then $\varphi \notin L^1$.

Aldroubi, Sun, Wang show:

Let $\varphi \in L^2$ and $\{t_k \varphi\}_{k \in \mathbb{Z}}$ be a Riesz basis of $S(\varphi)$, if $S(\varphi)$ is translation invariant, then $\varphi \notin L^1$.

Moreover:

Let $\varphi \in L^2$ and $\{t_k \varphi\}_{k \in \mathbb{Z}}$ be a Riesz basis of $S(\varphi)$, if $S(\varphi)$ is $\frac{1}{n}$-invariant for some $n \geq 2$ then

$$\forall \varepsilon > 0 \quad \int_{\mathbb{R}} |\varphi(x)|^2 |x|^{1+\varepsilon} dx = \infty$$

In particular, if $\varepsilon = 1 \Rightarrow \int_{\mathbb{R}} |\varphi(x) x|^2 dx = \infty$

Reminiscent of Balian-Low!!

Balian-Low : time-frequency concentration and non-redundancy are "incompatible" :

$\varphi \in L^2(\mathbb{R})$, $\Lambda \subseteq \mathbb{R}^2$ lattice $\{e^{2\pi i \eta x} \varphi(x-u) : (u,\eta) \in \Lambda\}$ Riesz basis

$$\Downarrow$$

$$\forall\, a,b \quad \underbrace{\left(\int (x-a)^2 |\varphi(x)|^2 dx\right)}_{T} \underbrace{\left(\int (\omega-b)^2 |\hat{\varphi}(\omega)|^2 d\omega\right)}_{F} = \infty$$

Note that T or F could be finite.

Amalgam Balian-Low : [BHW 95]

If $\{ e^{2\pi i \alpha k} \varphi(x-\beta j) : j,k \in \mathbb{Z} \}$ Riesz basis $\Rightarrow$

$$\varphi \notin S_0 := \{ f \in L^2 : Vf(t,\nu) \in L^1 \}$$

where $$Vf(t,\nu) = \int f(x) e^{-(x-t)^2} e^{2\pi i x\nu} dx$$

Rem: Both, BLT and ABLT are statements about smoothness, but neither one is stronger nor weaker.

Recall

Assume $\Lambda = \mathbb{Z} \times p\mathbb{Z}$, $p \in \mathbb{N}$, $\mu = (u, \eta)$:
$u, \eta \in \mathbb{Q}$

and $\pi(u, \eta)\varphi \in \mathcal{G}(\varphi, \mathbb{Z} \times p\mathbb{Z})$.

We have: $\pi(u, \eta)\varphi \in \mathcal{G}(\varphi, \mathbb{Z} \times p\mathbb{Z}) \iff$

$$Z(\pi(u, \eta)\varphi) \in Z\mathcal{G}(\varphi, \mathbb{Z} \times p\mathbb{Z}) =$$

$$= \overline{\operatorname{span}}\left\{ e^{2\pi i (p l x + k w)} Z\varphi(x, w),\ (k, l) \in \mathbb{Z}^2 \right\}$$

So $\pi(u,\eta)\varphi \in \mathcal{G}(\varphi, \mathbb{Z}\times p\mathbb{Z}) \Longleftrightarrow \exists\, \underset{\sim}{c} = (c_{k\ell}) \in \ell^2$

$$Z(\pi(u,\eta)\varphi) = e^{2\pi i \eta x} Z\varphi(x-u, \omega-\eta) =$$
$$= \sum c_{k\ell}\, e^{2\pi i (p\ell x + k\omega)} Z\varphi(x,\omega)$$
$$= h(x,\omega)\, Z\varphi(x,\omega) \quad \text{with}$$

$$h(x,\omega) = \sum_{k,l} c_{k\ell}\, e^{2\pi i (p\ell x + k\omega)}$$

which is $1/p$ - periodic in x and 1 - periodic in ω.

Obs: we used the fact that $\pi(\lambda)\varphi$ is a Riesz basis

With some 'busy work' we show that there exist integers R, M_1 and M_2: $\forall\ (x,\omega) \in \mathbb{R} \times \widehat{\mathbb{R}}$

$$\prod_{r=1}^{R} h(x + r\mu, \omega + r\eta) = e^{2\pi i (M_1 x - M_2 \omega)}$$

and if $Z\varphi$ is continuous $h(x,\omega)$ is continuous as well.

But this in turn implies (technical lemma)

- $\eta = k \cdot P$ and
- $\mu \in \mathbb{Z}$

CONTRADICTION - since $\mu = (\mu, \eta) \notin \Lambda = (\mathbb{Z}, P\mathbb{Z})$

So h can not exist if $\varphi \in S_0$!!

In general, we have the following:

Theorem: If (φ, Λ) is a Riesz basis for $\mathcal{G}(\varphi, \Lambda)$ with $\varphi \in S_0(\mathbb{R})$, and the density of Λ is rational, then $\pi(u, \eta)\varphi \notin \mathcal{G}(\varphi, \Lambda)$, for all $(u, \eta) \notin \Lambda$.

Note that this theorem generalizes the ABLT, since if (φ, Λ) is a Riesz basis of $L^2(\mathbb{R}) \Rightarrow \Lambda = \alpha\mathbb{Z} \times \beta\mathbb{Z}$ satisfies $\alpha \cdot \beta = 1 \in \mathbb{Q}$ and $\mathcal{G}(\varphi, \Lambda) = L^2(\mathbb{R}) \Rightarrow$ $\pi(u, \eta)\varphi \in \mathcal{G}(\varphi, \Lambda)$ $\forall (u, \eta) \in \mathbb{R} \times \widehat{\mathbb{R}}$ $\Rightarrow$ by our theorem $\varphi \notin S_0(\mathbb{R})$.

Examples:

It is straightforward to construct:

a) a discontinuous function φ such that

- $T_{1/2}\,\varphi \in \mathcal{G}(\varphi, \mathbb{Z} \times 3\mathbb{Z})$
- $\{\pi(\lambda)\varphi\}_{\lambda \in \mathbb{Z} \times 3\mathbb{Z}}$ is a Riesz basis for

Clearly φ cannot $\in S_0$!

b) a smooth function φ such that $T_{1/2}\varphi \in \mathcal{G}(\varphi, \mathbb{Z}\times 3\mathbb{Z})$ and so, $\{\pi(\lambda)\varphi\}_{\lambda\in(\mathbb{Z}\times 3\mathbb{Z})}$ is NOT a Riesz basis for $\mathcal{G}(\varphi, \mathbb{Z}\times 3\mathbb{Z})$

Characterization of extra invariance in SIS:

Proposition *Let S be a SIS. Then either S is translation-invariant, or there exists a maximum positive integer n such that S is $\frac{1}{n}\mathbb{Z}$-invariant.*

Further, if S is $\frac{1}{n}$ invariant and

$$S = S(\varphi) = \overline{\operatorname{span}}\{t_k\varphi : k \in \mathbb{Z}\}$$

$\{t_k\varphi\}_{k\in\mathbb{Z}}$ is a frame for S $\Longleftrightarrow$

$\{t_k\varphi^l\}_{k\in\mathbb{Z}}^{l=0,\dots,n-1}$ is a frame for S

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where φ^l is a 'cutoff' of φ :

Cutoff!

$B_k = B_k(n) := \bigcup_{j\in\mathbb{Z}} \big([k,k+1)+nj\big)$

$U_k := \{ f \in L^2 : \operatorname{supp} \hat{f} \subseteq B_k \}$. P_n projection,

$U_k = P_k(S)$ and $\varphi^k = P_k \varphi$.

Note that $f \in S \Leftrightarrow \hat{f} = m\,\hat{\varphi}$ with m $\mathbb{Z}$-periodic.

And $g \in U_k \Leftrightarrow \hat{g} = m_n \widehat{\varphi^k}$, m_n $n\mathbb{Z}$-periodic

Theorem 4.4. *If $S \subseteq L^2(\mathbb{R})$ is a SIS, then the following are equivalent.*

(a) *S is $\frac{1}{n}\mathbb{Z}$-invariant.*

(b) *$U_k \subseteq S$ for $k = 0, \ldots, n-1$.*

(c) *If $f \in S$, then $f^k = P_k f \in S$ for each $k = 0, \ldots, n-1$.*

Moreover, in case these hold we have that S is the orthogonal direct sum

$$S = U_0 \dot{\oplus} \ldots \dot{\oplus} U_{n-1},$$

with each U_k being a (possibly trivial) $\frac{1}{n}\mathbb{Z}$-invariant SIS.

One then has (after some work):

Corollary *Let $\varphi \in L^2(\mathbb{R})$ be given. If the SIS $S(\varphi)$ is $\frac{1}{n}\mathbb{Z}$-invariant for some $n > 1$, then $\widehat{\varphi}$ must vanish on a set of infinite Lebesgue measure. Furthermore, for each interval $I \subseteq \mathbb{R}$ of length n, we have that*

$$\left|\{\omega \in I : \widehat{\varphi}(\omega) = 0\}\right| \geq n\,|E_0| + (n-1)\,|E_1| \geq n-1,$$

where $E_0 = \{\omega \in [0,1) : G_\varphi(\omega) = 0\}$ and $E_1 = \{\omega \in [0,1) : G_\varphi(\omega) \neq 0\}$

Here $G_\varphi(\omega) := \sum_{k \in \mathbb{Z}} |\widehat{\varphi}(\omega + k)|^2$. In particular

If a nonzero function $\varphi \in L^2(\mathbb{R})$ has compact support, then $S(\varphi)$ is not $\frac{1}{n}\mathbb{Z}$-invariant for any $n > 1$.

Moreover, if $\varphi \in L^2(\mathbb{R})$ and $S(\varphi)$ is translation invariant, $|\operatorname{supp} \widehat{\varphi}| \leq 1$.

On going work on <u>Characterization</u> of extra invariance in $\mathcal{G}(\varphi, \mathbb{Z} \times p\mathbb{Z})$:

$\varphi \in L^2(\mathbb{R})$, $p, p' \in \mathbb{N}$, p' divides p.

$k = 0, \dots, p/p' - 1$;

$$B_k = B_k(p, p') \doteq \bigcup_{j \in \mathbb{Z}} \left(\frac{j}{p'} + \left[\frac{k}{p}, \frac{k+1}{p} \right] \right) \times \mathbb{R}$$

$$\mathcal{U}_k := \{ f \in L^2(\mathbb{R}) : Zf = Zg \cdot \chi_{B_k} \text{ for some } g \in \mathcal{G}(\varphi, \mathbb{Z} \times p\mathbb{Z}) \}.$$

We then have the following:

Theorem 29 (cf. Theorems 4.4 and 4.7 in [ACH+10]). *Let $\varphi \in L^2(\mathbb{R})$ and $P, P' \geqslant 1$ integers such that $P' \mid P$, that is, P' is a divisor of P. Then the following are equivalent.*

(a) *$\mathcal{G}(\varphi, \mathbb{Z} \times P\mathbb{Z})$ is invariant under $\pi(0, P')$.*
(b) *$U_k \subseteq \mathcal{G}(\varphi, \mathbb{Z} \times P\mathbb{Z})$ for $k = 0, \cdots, P/P' - 1$.*
(c) *$Z\varphi \cdot \chi_{B_k} \in Z[\mathcal{G}(\varphi, \mathbb{Z} \times P\mathbb{Z})]$ for $k = 0, \cdots, P/P' - 1$.*

Moreover in this case, $\mathcal{G}(\varphi, \mathbb{Z} \times P\mathbb{Z})$ is the orthogonal direct sum

$$\mathcal{G}(\varphi, \mathbb{Z} \times P\mathbb{Z}) = U_0 \oplus \cdots \oplus U_{P/P'-1}$$

with each U_k being a (possibly trivial) subspace of $\mathcal{G}(\varphi, \mathbb{Z} \times P\mathbb{Z})$ invariant under $\{\pi(k, P'\ell)\}_{k,\ell \in \mathbb{Z}}$.

collaborators:

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Thank You

very much !!

Technical Lemma:

If $h(x)$ is a function satisfying

$$e^{2\pi i Mx} = \prod_{r=1}^{R} h(x + r \cdot 0) = \left(h(x)\right)^R$$

* $h(x) \neq 0$.

If $h(x)$ is $\frac{1}{P}$-periodic, then

* RP divides M

$$h(x) = h\left(x + \frac{1}{P}\right) = e^{2\pi i \frac{M}{R}\left(x + \frac{1}{P}\right)} = e^{2\pi i \frac{M}{RP}} h(x).$$