On pointwise estimates involving sparse operators

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Sparse families and operators

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$$M^{\mathcal{D}} f(x) = \sup_{Q \ni x, Q \in \mathcal{D}} |f|_Q,$$

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- Proof: write $\Omega_k = \{x : M^{\mathcal{D}} f(x) > 2^{(n+1)k}\} = \cup_j Q_j^k$ and set $E_j^k = Q_j^k \setminus \Omega_{k+1}$. Then the claim holds with $\mathcal{S} = \{Q_j^k\}$.

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- A one-third trick: there are 3^n dyadic lattices $\mathscr{D}^{(j)}$ such that for every cube $Q \subset \mathbb{R}^n$, there is a cube $P \in \mathscr{D}^{(j)}$ for some j, containing Q and such that $|P| \leqslant 6^n |Q|$.

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- By the standard claim, for every $f \in L^1(\mathbb{R}^n)$, there are $\frac{1}{2}$-sparse families $\mathcal{S}_j \subset \mathscr{D}^{(j)}, j = 1, \ldots, 3^n$, such that
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- We will show that almost the same pointwise domination holds for Calderón-Zygmund operators T:

$$|Tf(x)| \leqslant C(n,T) \sum_{j=1}^{3^n} \sum_{Q \in \mathcal{S}_j} |f|_Q \chi_Q(x).$$

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- We say that T is an ω-Calderón-Zygmund operator if
  1. T is L^2 bounded;
  2. T is represented as

$$Tf(x) = \int_{\mathbb{R}^n} K(x,y) f(y) dy \quad \text{for all } x \notin \operatorname{supp} f;$$

  3. K satisfies the size condition $|K(x,y)| \leqslant \frac{C_K}{|x-y|^n}, x \neq y$;
  4. K satisfies the regularity condition

$$|K(x,y) - K(x',y)| + |K(y,x) - K(y,x')| \leqslant \omega\left(\frac{|x-x'|}{|x-y|}\right) \frac{1}{|x-y|^n}$$

for $|x-y| > 2|x-x'|$, where $\omega : [0,1] \to [0,\infty)$ is continuous, increasing, subadditive and $\omega(0) = 0$.

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- If $\mathcal{S}$ is a sparse family, then the operator

$$\mathcal{A}_{\mathcal{S}} f(x) = \sum_{Q \in \mathcal{S}} f_Q \chi_Q(x)$$

is called the sparse operator.

A very brief history

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$$\|T\|_{L^2(w)} \leqslant c(n,T)[w]_{A_2},$$

where $[w]_{A_2} = \sup_Q \left(\frac{1}{|Q|}\int_Q w\right)\left(\frac{1}{|Q|}\int_Q w^{-1}\right)$.

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$$\|Tf\|_X \leqslant c(n,T)\sup_{\mathscr{D},\mathcal{S}} \|\mathcal{A}_{\mathcal{S}}|f|\|_X.$$

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- The key idea behind of all approaches is an iteration:
  - iteration of the distribution function (rearrangement) of f with a substitution $f \to Tf$ (70's-80's);
  - iteration of f (pointwise, "a median decomposition") with a substitution $f \to Tf$;
  - iteration of Tf (M. Lacey).

Main steps of the proof

- The key recursive claim: there exist pairwise disjoint cubes $P_j \in \mathcal{D}(Q_0)$ such that $\sum_j |P_j| \leqslant \frac{1}{2}|Q_0|$ and for a.e. on Q_0,

$$|T(f\chi_{3Q_0})(x)|\chi_{Q_0} \leqslant c_n C_T |f|_{3Q_0} + \sum_j |T(f\chi_{3P_j})|\chi_{P_j}.$$

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- After iteration we obtain that there exists a $\frac{1}{2}$-sparse family $\mathcal{F} \subset \mathcal{D}(Q_0)$ such that

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- For arbitrary pairwise disjoint cubes $P_j \in \mathcal{D}(Q_0)$,

$$\begin{aligned} |T(f\chi_{3Q_0})|\chi_{Q_0} &\leqslant |T(f\chi_{3Q_0})|\chi_{Q_0\setminus\cup_j P_j} + \sum_j |T(f\chi_{3Q_0\setminus 3P_j})|\chi_{P_j} \\ &+ \sum_j |T(f\chi_{3P_j})|\chi_{P_j}. \end{aligned}$$

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- Hence, it suffices to find a set $E \subset Q_0$ and a covering of E by disjoint cubes $P_j \in \mathcal{D}(Q_0)$ such that
  1. $\sum_j |P_j| \leqslant \frac{1}{2}|Q_0|$;
  2. $|T(f\chi_{3Q_0})(x)| \leqslant c_n C_T |f|_{3Q_0}$ for a.e. $x \in Q_0 \setminus E$;
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- The following local "grand maximal truncated" operator

$$\mathcal{M}_{T,Q_0} f(x) = \sup_{P \ni x, P \subset Q_0} \operatorname*{ess\,sup}_{\xi \in P} |T(f\chi_{3Q_0 \setminus 3P})(\xi)|$$

  controls condition 3.

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- We have $\|\mathcal{M}_{T,Q_0}\|_{L^1 \to L^{1,\infty}} \leqslant \alpha_n C_T$ and

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  where c_n is such that $|E| \leqslant \frac{1}{2^{n+2}}|Q_0|$.
- Apply the Calderón-Zygmund decomposition to χ_E with $\lambda = \frac{1}{2^{n+1}}$. We obtain disjoint cubes $P_j \in \mathcal{D}(Q_0)$ such that $\frac{1}{2^{n+1}} < \frac{|P_j \cap E|}{|P_j|} \leqslant \frac{1}{2}$, which easily implies 1, 2 and 3.

Related remarks and questions

- The proof shows that if T is a sublinear operator of weak type $(1,1)$ and
$$\mathcal{M}_T f(x) = \sup_{Q \ni x} \operatorname*{ess\,sup}_{\xi \in Q} |T(f\chi_{\mathbb{R}^n \setminus 3Q})(\xi)|$$
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- Observe that the question whether the maximal singular integral operator $T_\Omega^\star$ is of weak type $(1,1)$ is still open.

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- In particular, we obtain the following result: if $\mu, \lambda \in A_p, 1 < p < \infty$, $\nu = (\mu/\lambda)^{1/p}$ and

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- This provides a quantitative form of the two-weighted bound due to S. Bloom (1985) and I. Holmes, M. Lacey and B. Wick (2015).

Some related "sparse domination" works

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Thank you for your attention!