Kakeya sets in three dimensions near the Wolff exponent

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Wolff Kakeya

Our goal is to show there exists a universal $\epsilon > 0$ so that any Kakeya set in $\mathbf{R}^3$ has Hausdorff dimension at least $\frac{5}{2} + \epsilon$.

We think of our Kakeya set as approximated by δ^{-2} many δ tubes with δ separated directions. Our goal is to show their union has measure at least $\delta^{\frac{1}{2}-\epsilon}$.

Wolff showed that the Hausdorff dimension is at least $\frac{5}{2}$. His key tool is the study of the hairbrush of a tube. If T is a tube, let $H(T)$ be the set of tubes which intersect T at angle approximately 1. Wolff showed that tubes of $H(T)$ are approximately disjoint.

Our takeaway from Wolff's argument is that for essentially each T in a near $\frac{5}{2}$ dimensional Kakeya set, we have $H(T)$ consisting of $\delta^{-\frac{3}{2}}$ tubes whose union essentially covers the Kakeya set.

K-Laba-Tao argument I: structure

In a paper with Laba and Tao, we showed that the upper Minkowski dimension (box counting) of Kakeya sets in $\mathbf{R}^3$ is at least $\frac{5}{2} + 10^{-10}$.

Our main idea was that Kakeya sets near dimension $\frac{5}{2}$ have strong structural properties with funny names.

Stickiness There are $\frac{1}{\delta}$ Fat tubes (of dimensions $\delta^{\frac{1}{2}} \times 1$) each containing $\frac{1}{\delta}$ many δ tubes. (We used Wolff's X-ray estimate in a way that essentially used that we were working with upper Minkowski dimension.)

Planyness At each δ-cube of the Kakeya set, there is a $\delta^{\frac{1}{2}}$ thickened plane containing all tubes of the set through that cube.

Graininess At each δ cube of the Kakeya set, there is a $\delta^{\frac{1}{2}} \times \delta^{\frac{1}{2}} \times \delta$ flat containing all nearby points of the Kakeya set.

K-Laba-Tao argument II: the Heisenberg group

The main idea of K-L-T: Even with stickiness/planyness/graininess, there is still an enemy. It is a $\frac{5}{2}$ dimensional Kakeya set which almost exists. "The Heisenberg group"

In $\mathbf{C}^3$, consider the set $Im(z) = Re(uw)$. This contains a 2 dimensional set of complex lines.

Of course, the real numbers contain no half-dimensional subring.

Moreover, the lines of the Heisenberg group are not all in different directions.

In KLT, we used that many lines would have to be in the same direction.

With Bourgain's discretized sum-product theorem, it became possible also to use that the reals don't have positive dimensional subrings.

BKT obsolete argument

Our initial idea was to mimic our argument after an old argument of Bourgain-K-Tao which shows that Kakeya sets in F_p^3 have dimension at least $\frac{5}{2} + \epsilon$.

Ironically, this argument is totally obsolete because Dvir has solved the Kakeya problem over F_p. Sometimes, however, obsolete arguments can be more useful than up-to-date ones because they are easier to mimic.

BKT idea: Take $H(L_1, L_2)$ for L_1, L_2 fixed lines and call its elements points. Now take $H(L_3, L_4)$ for L_3, L_4 lines and call its elements lines. A typical element of $H(L_3, L_4)$ intersects about $p^{\frac{1}{2}}$ lines of $H(L_1, L_2)$. Call such an intersection an incidence. Now we have an impossible point-line incidence configuration.

Why is it a point line configuration? It's related to the fact that $H(L_1, L_2, L_3)$ for any three lines in general position lie in one ruling of a doubly-ruled quadratic surface, the regulus.

Poor man's stickiness

We discuss the question of why we should expect any triples T_1, T_2, T_3 of δ-tubes in a Kakeya set to be in general position. The reason is an old idea of Wolff's that serves as a poor man's version of stickiness. If all tubes hit a δ^μ tube, then their union exceeds $\delta^{\frac{1}{2}}$.

Begin by looking in a particular hairbrush $H(T)$. If no pairs T_1, T_2 are in general position, then all tubes of $H(T)$ are within δ^μ of the plane spanned by T_1 and T_2. Since $H(T)$ covers the Kakeya set, this is true of all tubes of the Kakeya set. Conclusion: There are δ^{-5} triples (T, T_1, T_2) with T_1, T_2 skew in $H(T)$.

For T_1, T_2, T_3 to be in general position, it is enough that they be pairwise skew. We just run the same argument again finding $\delta^{-6.5}$ quartuples (T, T_1, T_2, T_3) with T_1, T_2, T_3 pairwise skew and in $H(T)$. This guarantees at least $\delta^{-.5}$ T in $H(T_1, T_2, T_3)$ for a typical triple (T_1, T_2, T_3).

Poor man's stickiness continued

Similarly δ^{-8} quintuples (T, T_1, T_2, T_3, T_4). But T_4 may be tangent to the regulus of T_1, T_2, T_3. In the extreme case, all T's may fill a regulus strip of density $\delta^{\frac{1}{2}}$. Incidentally, this induces strong planyness on T with a quadratic plane map.