

Summary of previous lecture

\mathcal{S} = space of subsolutions:

- smooth
- compactly supported in some $\Omega \times \mathbb{I}$
- with energy profile e

$$X = \overline{\mathcal{S}}_{L^{\infty}w^*}$$

X_S = subset of X of actual solutions

Theorem $X \setminus X_S$ is a set of first category.

Energy identity / inequality

$$\mathcal{R} \times \mathbb{I} = \mathcal{R} \times [0, +\infty]$$

$$\partial_t \frac{|v|^2}{2} + \operatorname{div} v \left(v - \left(\frac{|v|^2}{2} + p \right) \right) = 0 \quad (\leq 0)$$

Recall: $|v|^2 = v \cdot \mathbb{1}_\Omega$ energy profile

$$p = q - \frac{|v|^2}{n} = q - \frac{e}{n}$$

$$\partial_t \frac{e \mathbb{1}_\Omega}{2} + \operatorname{div} \left(v \left(q + e \left(\frac{1}{2} - \frac{1}{n} \right) \mathbb{1}_\Omega \right) \right) = 0 \quad (\leq 0)$$

Since v is supported in $\mathcal{R} \times [0, +\infty]$

$$v \cdot \mathbb{1}_\Omega = v$$

If we were able to produce solutions for which
 $q = \text{const}$, then

$$\text{div} (\sigma (q + c (\frac{1}{2} - \frac{1}{u}) \nabla u)) = \text{div} (\text{constant } \sigma) \\ = 0$$

and the entropy / energy
condition becomes

$$\begin{aligned} \partial_t e &= 0 \\ \text{or} \\ \partial_t e &\leq 0 \end{aligned}$$

Prismless solutions

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Note

$$\overline{U} = \begin{pmatrix} v_2 \otimes v_1 & v_1 \\ v_1 & 0 \end{pmatrix} - \begin{pmatrix} v_2 \otimes v_2 & v_2 \\ v_2 & 0 \end{pmatrix}$$

\hookrightarrow in the worse case

Recall: we must check that $\ker \overline{U} \neq \emptyset$
 Assume $n=2$ (otherwise $\exists \xi \in \text{span}(v_1, v_2)$ and $(\xi, 0) \in \ker \overline{U}$)

$$\overline{U} = \begin{pmatrix} v_1 \otimes v_1 - v_2 \otimes v_2 & v_1 - v_2 \\ v_1 - v_2 & 0 \end{pmatrix}$$

$$\text{Pick } \xi = (v_1 - v_2)^\perp \quad (\alpha, \beta)^\perp := (-\beta, \alpha)$$

$$\overline{U} + \begin{pmatrix} (v_1 - v_2)^\perp \\ \tau \end{pmatrix} = \begin{pmatrix} v_2 \otimes v_1 - (-v_2)^\perp - v_2 \otimes v_2 \cdot v_1^\perp + \tau(v_1 - v_2) \\ v_2 \end{pmatrix}$$

$$v_1 \otimes v_1 \cdot (-v_2)^\perp - v_2 \otimes v_2 \cdot v_1^\perp + \tau(v_1 - v_2)$$

$$= -\langle v_1, v_2^\perp \rangle v_1 - \langle v_2, v_1^\perp \rangle v_2 + \tau(v_1 - v_2)$$

$$\langle (a, b), (c, d)^\perp \rangle = \langle (a, b), (-d, c) \rangle = bc - ad$$

$$\langle (a, b)^\perp, (c, d) \rangle = -bc + ad$$

$$= \langle v_2, v_1^\perp \rangle (v_1 - v_2) + \tau (v_1 - v_2) = (\langle v_2, v_1^\perp \rangle + \tau) (v_1 - v_2)$$

Conclusion $((v_1 - v_2)^\perp, -\langle v_2, v_1^\perp \rangle)$ is in the kernel of ψ

This analysis suggests:

it is possible to solve

$$\begin{cases} \partial_t \sigma + \operatorname{div} \sigma = 0 \\ \operatorname{div} \sigma = 0 \end{cases} \quad \textcircled{+} \quad u = \sigma \otimes \sigma - \frac{|\sigma|^2}{n} \operatorname{Id}$$

using the "differentiable inclusions techniques"

A few details to add:

- 1) Adopt the potentials, since $U = \begin{pmatrix} u & \sigma \\ \sigma & 0 \end{pmatrix}$ is trace-free.
- 2) Run the rest of the program

Consequence :

We can construct solutions which satisfy the local energy identity / inequality

Problem : The initial data

The real energy inequality in the sense of distributions is

$$\int_0^\infty \int_{\frac{1}{2}}^\infty |u|^2 \varphi \, dx dt + \int_0^\infty \int_0^\infty \left(\frac{|u|^2}{2} + \rho \right) u \cdot \nabla \varphi \, dx dt$$

$$= - \int \frac{|u|^2}{2} (x, 0) \, dx$$

(\Leftarrow)

initial data

To pass from

$$\partial_t \frac{|v|^2}{2} + \operatorname{div}\left(\left(\frac{|v|^2}{2} + \rho\right)v\right) = 0$$

(\leq)

in the set $\mathcal{D} \times]0, +\infty[$

to the previous identity we need

$$v(\cdot, t) \rightarrow v(\cdot, 0) \quad \underline{\text{strongly in } L^2}$$

This certainly cannot be achieved with subsequences
which are compactly supported, since in this
case $v(\cdot, 0) = 0$

Adopted subsolutions

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$$\Omega \times [0, +\infty[$$

$(v_0, u_0, 0)$ is

-) smooth in $\mathbb{R}^n \times [0, +\infty[$ BUT NOT in $\mathbb{R}^n \times]0, +\infty[$
-) solves the linear relation

$$\begin{cases} \partial_t v_0 + \Delta v_0 = 0 \\ \Delta u_0 v_0 = 0 \end{cases}$$

-) satisfies $v_0 \otimes v_0 - u_0 < \frac{\epsilon}{u}$. Id in $\Omega \times [0, +\infty[$

-) vanishes outside Ω

-) $v_0(\cdot, t) \rightarrow v_0(\cdot, 0)$ weakly in L^2

and $\int |v_0|^2(x, 0) dx = \int_{\Omega} e(x, 0) dx$

Theorem

If you have an adopted subsolution with energy profile $e(x,t) = e(t)$ s.t. $e' \leq 0$, then \exists infinitely many solutions of incompressible Euler with initial data $v_0(\cdot, 0)$ satisfying the local energy inequality.

"Proof" Run the pressureless iteration

$$X_S = \text{space of actual solutions } (v, \rho) \in X_S$$

$$\text{Already observed: } \partial_t \frac{|v|^2}{2} + \operatorname{div} v (v \cdot \frac{|v|^2}{2} + \rho)) \leq 0$$

Note also: 1) $v(\cdot, t) \rightarrow v_0(\cdot, 0)$ weakly in L^2

$$2) \int_{\Omega} |\psi(x,t)|^2 dx = \int_{\Omega} e(x,t) dx = e(t) |\Omega|$$

$$3) \int_{\Omega} |\psi_0(x,0)|^2 dx = e(0) |\Omega|$$

$$\text{Since } e(t) \rightarrow e(0) \quad \psi(\cdot, t) \rightarrow \psi_0(\cdot, 0)$$

STRONGLY

problem: construct adapted sub-solutions

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Proposition

For any Ω and any e there are adopted sub-solutions !

"Proof" Start with the (non-adopted) subsolution $(0, 0, 0)$

Run the "explicit" iteration

$$(v_0, u_0, q_0) \rightarrow (v_1, u_1, q_1) \rightarrow \dots$$

in $\Omega \times]-1, 1[$

BUT at each step perfect (v_k, u_k, q_k) only

$$\text{in } \Omega \times \left] -\frac{1}{2^k}, \frac{1}{2^k} \right]$$

Note $\int |v_k|^2(x, t) dx \leq |\Omega| e(t)$

$$\text{and } \int |v_k|^2(x, \frac{1}{2^k}) dx = |\Omega| e\left(\frac{1}{2^k}\right) + o(1)$$

$$= |\Omega| e(0) + o(1)$$

We would be finished if we could upgrade

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"strong space-time convergence"

to $C(\mathbb{J}^{-1}, \mathcal{L}, L^2_{\text{strong}})$ convergence

possible by considering again a suitable Banach category
argument: v is in $V = C(\mathbb{J}^{-1}, \mathcal{L}, L^2_{\text{weak}})$

$$I = \inf_t \int_{\Omega} (|v(x,t)|^2 - e(x,t)) dx$$

is a lower semicontinuous function

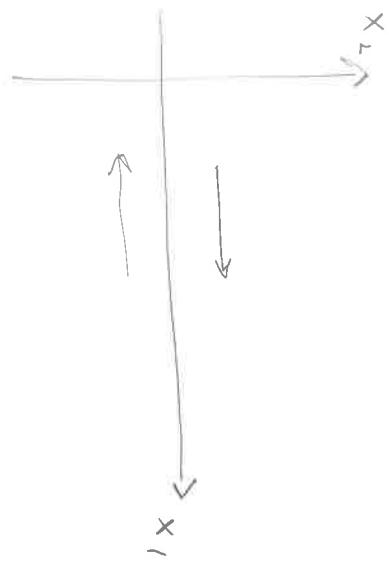
Hence Barre-I

Its points of continuity are needed!

D

Lá'szle's trick

Consider the shear flow instead of data

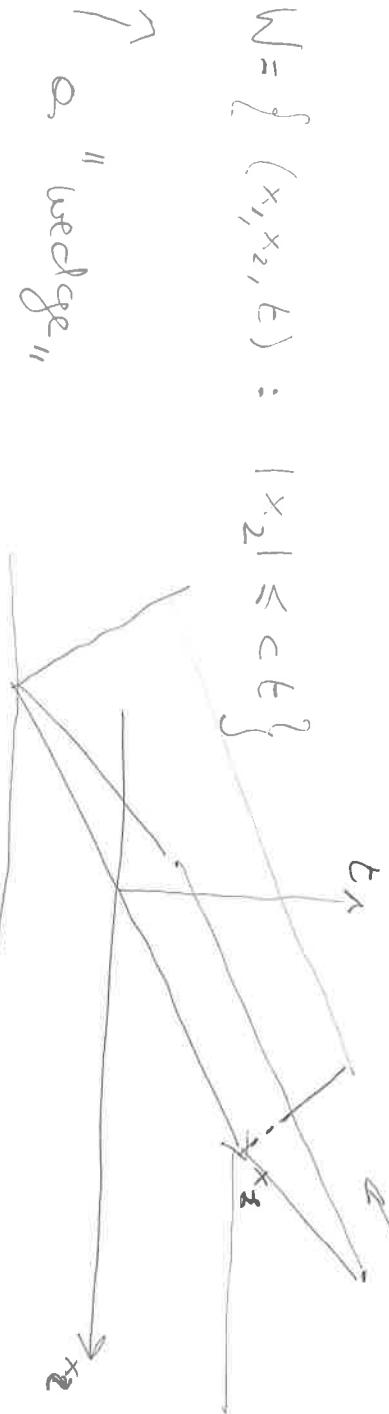


$$v_0(\cdot, 0) = (1, 0) \text{ for } x_2 > 0$$

$$v_0(\cdot, 0) = (-1, 0) \text{ for } x_2 < 0$$

Instead of looking for adopted subsolutions in a domain $\mathcal{Q} \times]0, +\infty[$, look for a subsolution W

in $W = \{(x_1, x_2, t) : |x_2| \leq ct\}$



Now $u_0 \equiv 0$ and $v_0(x, t) = v_0(x, 0)$ OUTSIDE W

Proposition (Serebrovskii)

An adopted substation can be constructed in W solving the Burgers' equation (with a "rarefaction wave")

Sasha's trick: instead of looking for smooth substation
look for piecewise constant substation !

Lemma: There are adopted piecewise constant substation
in W

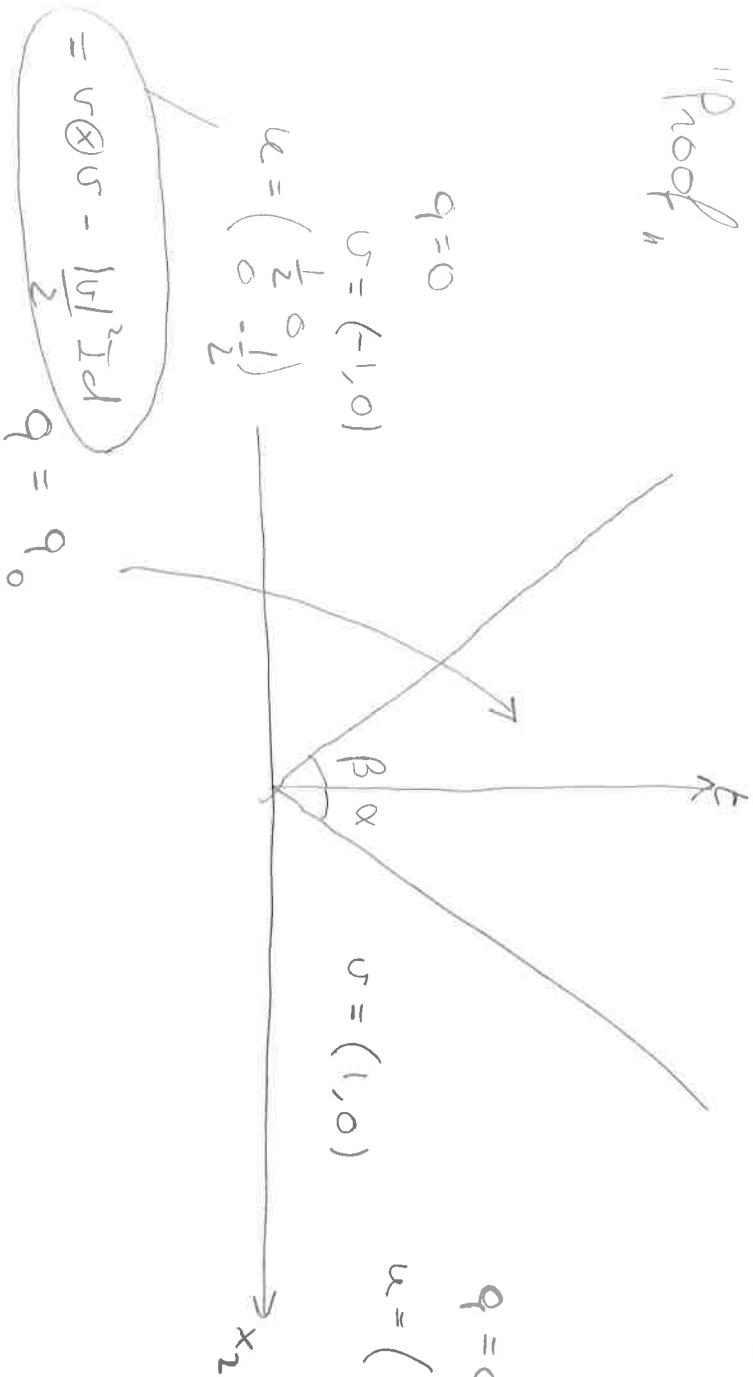
Ad vanage: The problem is "purely algebraic"

"Proof"

$$c = 1$$

other choices
possible
but $c(0) = 1$
is necessary)

$$\begin{aligned}q &= 0 \\v &= (-1, 0) \\u &= \left(\frac{1}{2}, 0\right) \\u &= \left(\frac{1}{2}, -\frac{1}{2}\right)\end{aligned}$$



$$= v \otimes v - \frac{|v|^2}{2} Id$$

$$q = q_0$$

$$v = (a, b)$$

$$u = \begin{pmatrix} c & d \\ d & -c \end{pmatrix}$$

Number of parameters: 7, including the angles α, β

6 equations and 1 Max x inequality

↑
soft constraint

$$\boxed{\begin{array}{l} \lambda_1 + \lambda_2 > 0 \\ \lambda_1, \lambda_2 > 0 \end{array}}$$

↑
hard constraint

□

(3)

Compressible Isentropic Euler

$$\left\{ \begin{array}{l} \partial_t \rho + \operatorname{div}_x (\rho \boldsymbol{v}) = 0 \\ \partial_t (\rho \boldsymbol{v}) + \operatorname{div}_x (\rho \boldsymbol{v} \otimes \boldsymbol{v}) + \nabla_x [\rho (\rho)] = 0 \end{array} \right.$$

Energy inequality (a dissipation condition)

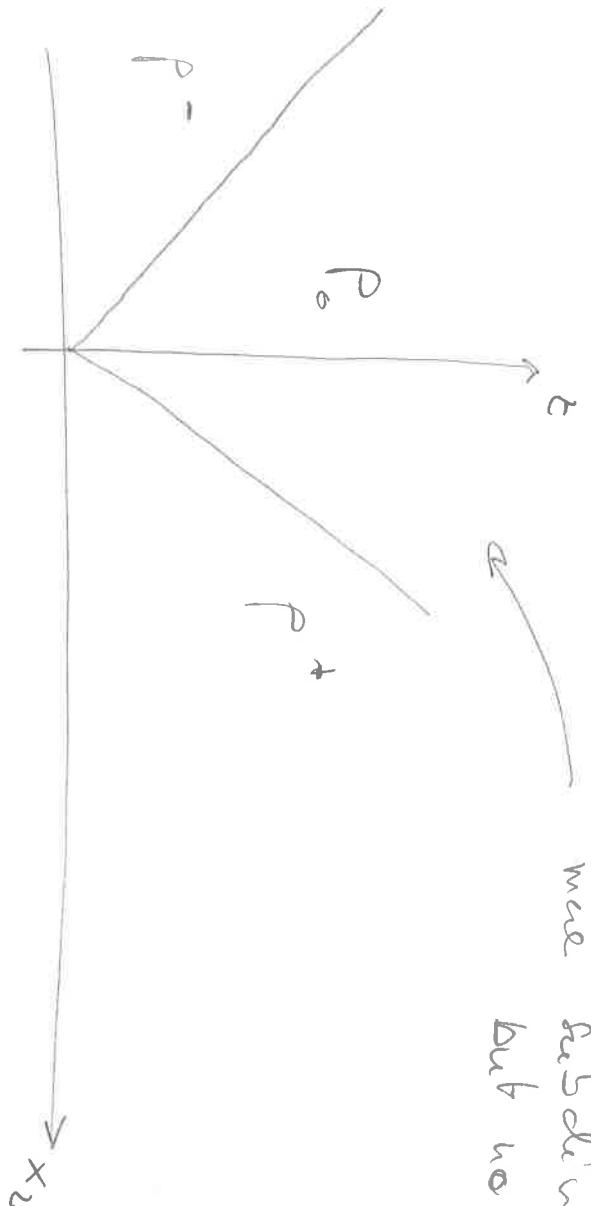
$$\partial_t \left(\rho \varepsilon(\rho) + \rho \frac{|\boldsymbol{v}|^2}{2} \right) + \operatorname{div}_x \left[\left(\rho \varepsilon(\rho) + \rho \frac{|\boldsymbol{v}|^2}{2} + \rho (\rho) \right) \boldsymbol{v} \right] \leq 0$$

Riemann data

$$(\boldsymbol{v}(\cdot, 0), \rho(\cdot, 0)) = \begin{cases} (\boldsymbol{v}^+, \rho^+) & x_2 > 0 \\ (\boldsymbol{v}^-, \rho^-) & x_2 < 0 \end{cases}$$

Few sub-solutions

more subdivisions possible
but not needed!



Subsolution:

$$(\bar{\rho}, \bar{v}, \bar{u}) = (\rho^+, v^+, u^+) \mathbb{1}_{\rho^+} + (\rho^-, v^-, u^-) \mathbb{1}_{\rho^-}$$

$$+ (\rho_0, v_0, u_0) \mathbb{1}_{\rho}$$

↑
constant

C = "energy level in ρ "

(and) has:

$$1) \quad \partial_t \bar{\rho} + \operatorname{div}_x (\bar{\rho} \bar{u}) = 0$$

Idea: "presureless convex integration"

adds a divergence-free field \underline{v} supported in ρ_0 :

$$v = \underline{v} + \underline{v}$$
 and thus

$$\partial_t \bar{\rho} + \operatorname{div}_x (\bar{\rho} \underline{v}) = 0$$

Mass balance OK

$$2) \quad \partial_t (\bar{\rho} \bar{u}) + \operatorname{div}_x (\bar{\rho} \bar{u}) + \nabla_x (\rho(\bar{\rho}) + \frac{1}{2} (\rho c_0)^2 \bar{\rho} + \rho^2 \frac{|\underline{v}|^2}{2} \bar{\rho} + \rho^2 \frac{|u|^2}{2} \bar{\rho}) = 0$$

Idea: "presymmetry under integration" adds

$\underline{\sigma}$ and $\underline{\omega}$ so that

- $(\underline{\sigma}, \underline{\omega})$ is supported in Ω .

$$\partial_t \underline{\sigma} + d \omega \underline{\omega} = 0$$

$$\bullet) (\bar{\sigma} \underline{\sigma}) \otimes (\bar{\sigma} + \underline{\sigma}) - (\bar{\sigma} + \underline{\sigma}) = \frac{1}{2} \bar{\sigma}^2 \text{Id} = \frac{C_0}{2} \text{Id}$$

So if $\sigma = \bar{\sigma} + \underline{\sigma}$ we get

$$\partial_t (\bar{\rho} \sigma) + d \omega (\bar{\rho} (\sigma \otimes \sigma - \frac{|\sigma|^2}{2} \text{Id})) + \nabla \left[\rho (\bar{\rho}) + \bar{\rho} \frac{|\sigma|^2}{2} \right] = 0$$

$$\Rightarrow \partial_t (\bar{\rho} \sigma) + d \omega (\bar{\rho} [\sigma \otimes \sigma] + \nabla \cdot \rho (\bar{\rho})) = 0$$

Beken of Riemannian



3) Energy condition

(5)

$$\partial_t (\bar{\rho} \varepsilon(\bar{\rho})) + \partial_x \sigma_x ((\bar{\rho} \varepsilon(\bar{\rho}) + \rho(\bar{\rho})) \bar{\sigma})$$

$$+ \partial_t \left(- \rho^+ \frac{|\sigma^+|^2}{2} \nabla \rho^+ + \bar{\rho} \frac{|\sigma^-|^2}{2} \nabla \rho^- + \rho_0 C_0 \nabla \rho_0 \right)$$

$$+ \text{div}_x \left[\left(\rho^+ \frac{|\sigma^+|^2}{2} \nabla \rho^+ + \rho^- \frac{|\sigma^-|^2}{2} \nabla \rho^- + \rho_0 C_0 \nabla \rho_0 \right) \bar{\sigma} \right] \leq 0$$

Idea : \textcircled{A} is the kinetic energy density of the sol.
 So after running "pressureless convex integration",

$$\int_t \left(\bar{\rho} \varepsilon(\bar{\rho}) + \bar{\rho} \frac{|\sigma|^2}{2} \right) + \text{div}_x \left[(\bar{\rho} \varepsilon(\bar{\rho}) + \rho(\bar{\rho}) + \frac{|\sigma|^2}{2} \bar{\rho}) \bar{\sigma} \right] \leq 0$$

$$\text{But } \sigma = \bar{\sigma} + \underline{\sigma}$$

$\underline{\sigma}$ supplied in ρ_0 , where \textcircled{E} is constant.

and σ diverges-free:

$$f_t \left(\bar{\rho} \varepsilon (\bar{\rho}) + \bar{\rho} \frac{|\sigma|^2}{2} \right) + d_1 \sigma_x \left[(\bar{\rho} \varepsilon (\bar{\rho}) + \rho(\bar{\rho}) + \frac{|\sigma|^2}{2} \bar{\rho}) \sigma \right] < 0$$

4) final condition:

$$v_0 \otimes v_0 - u_0 < \frac{C_0}{2} \text{Id}$$



"subsolow condition" to be able to run the iteration in P_0 !

Theorem

If there is a "fan-subsolution", then

there is an uniquely way admissible solutions of
isentropic Euler with the corresponding data

$$(v^{\pm}, \rho^{\pm}) = \begin{cases} (v^+, \rho^+) & x_2 > 0 \\ (v^-, \rho^-) & x_2 < 0 \end{cases}$$

problem

1) Construct subsolutions

2) Get one subsolution with a Riemann-data

which is the blow-up of a smooth compression wave.

Both 1) and 2) find a solution in N real unknowns
of a (complicated!) set of algebraic equations and
inequalities

→ A scaling $\phi(r)$:

Unknowns: $(\sigma^+, \sigma^-, \sigma_0)$

6 numbers

(ρ^+, ρ^-, ρ_0)

3 numbers

α_0 2 numbers

angles of the wedge 2 numbers

13

Equations:

Balance of mass 1 PDE
Balance of momentum 2 PDEs
Boundary conditions 2 interfaces → 4 eq.
→ 2 eq. } TOT 6

Indicities

Energy ineq.
Subsaturation condition
2 interfaces → 2 ineq. } TOT 4
2 ineq. } TOT 4

10 conditions on 13 unknowns

2) Campaña wrote and Nor

1 oddi Norad eq.

1 oddi Norad ineq.