

The (incompressible) Euler equations as a differential inclusion

$$\sigma : \mathbb{R}^n \times \mathbb{I} \rightarrow \mathbb{R}^n \quad \rho : \mathbb{R}^n \times \mathbb{I} \rightarrow \mathbb{R}$$

$$\left\{ \begin{array}{l} \partial_t \sigma + \operatorname{div} \sigma (\sigma \otimes \sigma) + \nabla \rho = 0 \\ \operatorname{div} \sigma = 0 \end{array} \right. \quad (1)$$

$$\left\{ \begin{array}{l} \operatorname{div} \sigma = 0 \\ \operatorname{curl} \sigma = 0 \end{array} \right. \quad (2)$$

Rewrite (1) as

$$\partial_t \sigma + \operatorname{div} \sigma (\sigma \otimes \sigma - \underbrace{\frac{|\sigma|^2}{n} \mathbb{I}_d}_{\text{Symmetric and trace-free}}) + \nabla \left(\rho + \underbrace{\frac{|\sigma|^2}{n}}_{=: q} \right) = 0$$

Symmetric and
trace-free

$$=: u \in \operatorname{Sym}^{n \times n}_0$$

(1) - (2) equivalent to

$$\left\{ \begin{array}{l} \partial_t u + \operatorname{div} u + \nabla q = 0 \\ \operatorname{div} u = 0 \end{array} \right.$$

(LR) \oplus

$$(AC) \quad u = \sigma \otimes \sigma - \frac{|\sigma|^2}{n} \mathbb{I}_d$$

①

Set further

$$U := \begin{pmatrix} u+q \text{Id} & v \\ v & 0 \end{pmatrix}$$

$$\mathcal{K} := \left\{ (u, v, q) : u = v \otimes v - \frac{|v|^2}{n} \text{Id} \right\}$$

$$(1) - (2) \iff (LR) + (\text{Ac}) \iff \text{div}_{x,t}^{\omega} U = 0 \oplus U \in \mathcal{K} \text{ a.e.}$$

(Euler)

Analogy with differential induction: $\nabla \alpha \in \mathcal{K}$, that is

$$\text{curl } H = 0 \oplus H \in \mathcal{K}$$

In fact there is some kind of "duality"

$$\mathcal{K}_e := \left\{ u = v \otimes v - \frac{|v|^2}{n} \text{Id} \text{ and } |v|^2 = e \right\}$$

↑ a positive constant (for the moment!)

Tartan's wave cone analysis

$\bar{U} = (\bar{G}, \bar{\omega}, \bar{q}) \in \mathbb{R}^n \times \text{Sym}_{nxn} \times \mathbb{R}$ belongs to the cone Λ
of the linear relative if $\exists (\xi, \varepsilon) \neq 0$ s.t.
 \underline{LR}

$$\bar{U} \propto (x \cdot \xi + t\varepsilon)$$

solves LR $\forall \alpha \in C^1$

Fowler: $\bar{U} e^{i\lambda(\xi, \varepsilon) \cdot (x, t)}$ solves (LR)

$$d\lambda \sigma \left(\bar{U} e^{i\lambda(\xi, \varepsilon)} \right) = i\lambda \bar{U} \cdot \left(\frac{\xi}{\varepsilon} \right) e^{i\lambda(\xi, \varepsilon)}$$

So $\bar{U} \in \Lambda \Leftrightarrow k_{\mathbb{C}} \bar{U} \neq \{0\}$

(2)

localized plane waves $(\bar{U}, \bar{u}, \bar{\sigma})$

Proposition Let $\bar{U} \in \Lambda$ with $\bar{U} \neq 0$. Let $\sigma = [-\bar{U}, \bar{U}]$

$\forall \varepsilon > 0 \quad \exists \quad U^\varepsilon \in C_c^\infty(B_1(0))$ "space-time convergence free s.t.
 $\|(\sigma, \alpha, \eta)\|$

o.)

$$\text{dist}(U(x, t), \sigma) < \varepsilon$$

$$U(x, t) \in B_1$$



b.)

$$\int |U|(x, t) dx dt \geq \alpha |\bar{U}|$$



$$-\bar{U}$$

positive dimensional constant

c) $\bigcup_{n+1, n+1} = 0$

$$\left(U = \begin{pmatrix} u + \varphi \text{Id} & v \\ v & 0 \end{pmatrix} \right)$$

"Proof"

$$U(x,t) = \overline{U} \sin(\lambda(\xi, \tau) \cdot (x, t)) \quad (\xi, \tau) \in \ker(\overline{U})$$

$$\int_{B_2} |v| = |\bar{v}| \int_{B_1} |\sin(\lambda(\xi, \tau) \cdot (x, t))| dx dt \geq \beta |\bar{v}|$$

But v is not compactly supported!

φ cut-off function. Problem: φv does not solve (LR)
otherwise.

Look for potentials:

\mathcal{L} linear diff. operator s.t. $\mathcal{L}(E)$ is

- symmetric
- divergence-free

Lemm

$$E_{ij}^{k\ell} \in C^\infty(\mathbb{R}^{n+1})$$

skew-symmetric in ij and $k\ell$

$$\mathcal{L}(E) = \frac{1}{2} \sum_{k\ell} \mathfrak{J}_{k\ell}^2 (E_{kj}^{i\ell} + E_{ki}^{j\ell})$$

gives a potential.

Proof

$$U = \mathcal{L}(E) : U_{ij} = U_{ji}$$

obvious!

$$\sum_i \mathfrak{J}_{ij} U_{ij} = \frac{1}{2} \sum_{k,\ell,i} \mathfrak{J}_k (\mathfrak{J}_{i\ell}^2 E_{kj}^{i\ell})$$

= 0

$$+ \frac{1}{2} \sum_{k,\ell,i} \mathfrak{J}_\ell (\mathfrak{J}_{ik}^2 E_{kj}^{i\ell}) = 0$$

□

"Proof of Proposition u" continued

Find a potential for \bar{U} s.t. $\lambda(\xi, \bar{z}) \cdot (x, t)$:

$$E(x, t) = \bar{E} - \frac{1}{\lambda^2} \sin(\lambda(\xi, \bar{z}) \cdot (x, t))$$

$$\begin{aligned} U := \mathcal{L}(\varphi E) &= \varphi \mathcal{L}(E) + O\left(\frac{1}{\lambda}\right) \\ &= \varphi \bar{U}^{S_n} + O\left(\frac{1}{\lambda}\right) \end{aligned}$$

$0 \leq \varphi \leq 1$
cut off

$$\int |\bar{U}| \geq |\bar{U}| \int |\varphi| d\mu - O\left(\frac{1}{\lambda}\right)$$

$$\geq \frac{|\bar{U}|}{2} \int |\varphi| d\mu$$

Closed & Borel

$$\varphi \bar{U} \sin(\cdot) \in \sigma = [-\bar{U}, \bar{U}]$$

Choose λ large: $\text{dist}(U, \sigma) < \varepsilon$

Λ - Convex hull and Subsolutions

$$K_e \times [-1, 1] = \{ (u, v, q) : |q| \leq 1, \quad u = v \otimes v - \frac{|v|^2}{h} \text{Id}$$

$$|v|^2 = e \}$$

Def. The Λ -Convex hull of $K_e \times [-1, 1]$ is given

by applying iteratively the following construction:

$$\tilde{\mathcal{M}}^{(1)}_e = \{ \lambda U_1 + (1-\lambda)U_2 : U_1, U_2 \in K_e \times [-1, 1], \\ U_2 - U_1 \in \Lambda \}$$

$$\mathcal{M}_e^{(2)} = \{ \lambda U_1 + (1-\lambda)U_2 : U_1, U_2 \in \tilde{\mathcal{M}}^{(1)}_e, \quad U_2 - U_1 \in \Lambda \}$$

$$\vdots$$

$$\mathcal{M}_e = \bigcup_{k \geq 1} \mathcal{M}_e^{(k)}$$

Remark

$$\mathcal{U}_e = \text{ch}(K_e) \times [-1, 1]$$

classical convex hull

$$\text{in } 2d, \text{ in } nd = \mathcal{U}_e^{(k(n))}$$

Lemma

$$\mathcal{U}_e = \mathcal{U}_e^{(1)} = \left\{ (v, u, q) : v \otimes v - u \leq \frac{e}{n} \text{Id}, |q| \leq 1 \right\}$$

"Proof"

$$\mathcal{U}_e \subset \{(v, u, q) : |q| \leq 1 \text{ and}$$

$$v \otimes v - u \leq \frac{e}{n} \text{Id}\} = S$$

Jensen's inequality!

) A bit more work: The extremal points
of S are $K_e \times \{-1, 1\}$

Def. (Subsolutions)

Let $e : \Omega \times \mathbb{I} \rightarrow \mathbb{R}^+$ be smooth

$$\mathbb{R}^n \times \mathbb{R}$$

A triple $(\bar{v}, \bar{u}, \bar{q}) \in C^\infty_c(\Omega \times \mathbb{I})$ is a subsolution of incompressible Euler with energy profile e if

$$(1) \quad \begin{cases} \partial_t \bar{v} + \operatorname{div} \bar{u} + \nabla \bar{q} = 0 \\ \operatorname{div} \bar{u} = 0 \end{cases}$$

$$(2) \quad |\bar{q}| < 1$$

$$(3) \quad \bar{v} \otimes \bar{u} - \bar{u} < \frac{e}{n} \operatorname{Id}$$

note: $(\bar{v}, \bar{u}, \bar{q})(x, t)$

$e(x, t)$

The iteration

Goal : start with (v_0, u_0, q_0) subsoletwa, for instance $(0, 0, 0)$,

It solves

$$\left\{ \begin{array}{l} A: \partial_t v_0 + \operatorname{div} u_0 + \nabla q_0 = 0 \\ \operatorname{div} u_0 = 0 \end{array} \right.$$

$$B: (v_0, u_0, q_0) \in \mathcal{V}_0$$

C it is commonly supposed in some $\Omega \times T$
use the localized plane waves to produce

(v_1, u_1, q_1) satisfying A and B and C

with

$$\int \text{dist}((\sigma_i, u_i, q_i), K_e) < (1-\delta) \int \text{dist}((\sigma_0, u_0, q_0), K_e)$$

↑

& positive geometric constant

Upshot:

Step 1

Locating on a small ball

We can observe (σ_0, u_0, q_0) constantly

equal to $(\tilde{\sigma}, \tilde{u}, \tilde{q})$

Step 2 △ convexity (quasimodulus + some work)

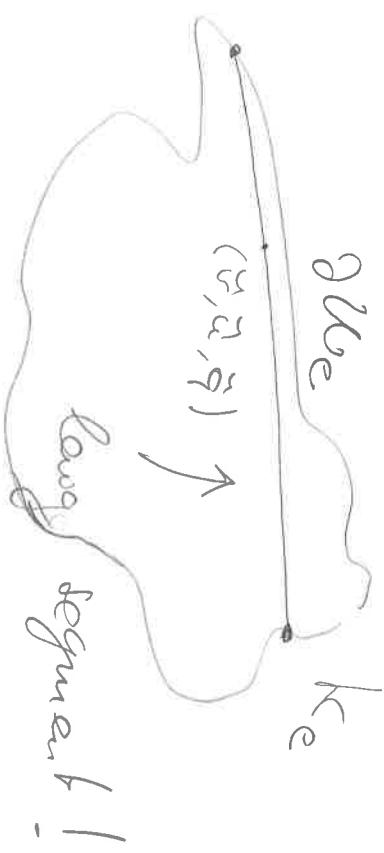
$$(\tilde{\sigma}, \tilde{u}, \tilde{q}) = \left(\sum_{i=1}^{N_1} \mu_i (\sigma_i, u_i, q_i) \right)$$

geometric

$$(v_i, u_i, q_i) \in K_e \times [-1, 1]$$

$$U := (v_i - v_j, u_i - u_j, q_i - q_j) \in \Lambda$$

Note: two k_e must be large, of the order of $\text{dist}((\bar{v}, \bar{u}, \bar{q}), K_e)$



Step 2

Add a localized plane wave in
the long direction and get closer to K_e !

$$(\bar{v}, \bar{u}, \bar{q}) + \varphi \mu U \sin(\lambda) + O\left(\frac{1}{\lambda}\right)$$

(3)

Step 4 Covering argument

Back to the iteration

$$(v_0, u_0, q_0) \rightarrow (v_1, u_1, q_1) \rightarrow (v_2, u_2, q_2) \rightarrow$$

- (v_k, u_k, q_k) c.s. in $\Omega \times \mathbb{I}$

- May solve the linear eq. on the whole space-time

$$\int_{\Omega \times \mathbb{I}} \text{dist}((v_k, u_k, q_k), K_e) \leq (1-\delta)^k \rightarrow 0$$

$K \rightarrow +\infty$

(v_k, u_k, q_k) converges weakly* in L^∞ (up to subseq.) to (v, u, q)

Note: • $(v, u, q) \in L^\infty$

- (v, u, q) is supported in $\Omega \times T$
- (v, u, q) solves $L.R.$ in $\mathbb{R}^n \times \mathbb{R}$

IF we had strong convergence:

$$(v, u, q)(x, t) \in K_c(x, t) \quad \text{a.e. } (x, t)$$

HENCE

$$u = v \otimes v - \frac{\|v\|^2}{n} \text{Id} \quad \left. \right\} \text{in } \Omega$$

and $\|u\|^2 = c$

BUT

$$0 = u = v \otimes v - \frac{\|u\|^2}{n} \text{Id} = 0$$

outside Ω

Conclusion

Theorem

$(v, \rho) = (v, q - \frac{|v|^2}{n})$ is a natural asymptoty

Supported solution of "incompromisal Evar"

Problem:

justify the strong convergence

Strong convergence

(1)

Millett - Sverdrup

The frequencies in the spectrum grow very fast

The approximating sequence behaves like

a Riemann Fourier series

(2)

Bowe Category argument.

(3)

Baire Category Argument

Ω, I, e fixed

$\mathcal{S} := \{ (v, u, q) \text{ smooth subsolutions, compactly supported in } \Omega \times I,$

with energy profile $e \}$

$X := \overline{\mathcal{S}}_{L^\infty-w^*} = \text{weak-* closure in } L^\infty \text{ of } \mathcal{S}$

Lemma X is bounded in L^∞ : the weak-* topology on X

is metrisable by some distance d . (X, d) is a complete
metric space

Def. $X_s \subset X$ is the space of "solutions":

$(v, u, q) \in X$ s.t. $(v, u, q)(x, t) \in K_{e(x, t)}$

a.e. $(x, t) \in \Omega \times I$

for a.e. $(x, t) \in \Omega \times I$

Theorem $X \setminus X_S$ is a set of first category.

Baire $\Rightarrow X_S$ is dense and so not empty

"Proof of the Theorem"

Lemma $\text{id} : (X, d) \rightarrow (X, \|\cdot\|_{L^1})$ is a Baire-1 map

(pointwise limit of continuous maps)

Proof $\text{id} * \varphi_\varepsilon : g \mapsto g * \varphi_\varepsilon$ is continuous

$\text{id} * \varphi_\varepsilon(g) \rightarrow g$ in L^1

□

Baire \Rightarrow the points of discontinuity of id is a set of first category

②

(3)

Claim $X \setminus X_s \subset$ discontinuities of $\text{id}: (X, \mathcal{L}) \rightarrow (X, \mathbb{M}_{\mathcal{L}'})$

Assume (v, u, q) is a point of continuity.

Let $(v_k, u_k, q_k) \in \mathcal{S}$ be s.t. $(v_k, u_k, q_k) \xrightarrow{*} (v, u, q)$

(recall: $X = \overline{\mathcal{S}}^{L^\infty\text{-weak*}}$)

Continuity $\Rightarrow \text{id} \Rightarrow (v_k, u_k, q_k) \rightarrow (v, u, q)$ strongly in \mathcal{L}'

Let $\Sigma := \{(x, t) : \text{dist}((v, u, q)(x, t), K_{e(x, t)}) > 0\}$

if $|\Sigma| > 0$, then $|\{(x, t) : \text{dist}(\quad, K_{e(x, t)}) > 2y\}| > 2y$ for some y .

$$\mathcal{D}_k := \{(x, t) : \text{dist}((v_k, u_k, q_k), K_e) > \gamma\}$$

Strong convergence: $|D_k| > \gamma$

Use the localized perturbations to construct

$$(\tilde{v}_k, \tilde{u}_k, \tilde{q}_k) \text{ with}$$

$$\cdot \int \text{dist}((\tilde{v}_k, \tilde{u}_k, \tilde{q}_k), K_e) \leq (1 - \delta) \int \text{dist}((v_k, u_k, q_k), K_e)$$

Observe: frequencies of oscillations might can be arranged

so that

$$\|(\tilde{v}_k, \tilde{u}_k, \tilde{q}_k) - (v_k, u_k, q_k)\|_{L^\infty} \leq \frac{1}{k}$$

Conclude

$$(\bar{v}_k, \bar{u}_k, \bar{q}_k) \xrightarrow{*} (\bar{v}, \bar{u}, \bar{q})$$

$$\lim_{K \rightarrow \infty} \int \text{dist}((\bar{v}_k, \bar{u}_k, \bar{q}_k), K_e) \leq (1-\delta) \int \text{dist}((\bar{v}, \bar{u}, \bar{q}), K_e)$$

So $(\bar{v}_k, \bar{u}_k, \bar{q}_k)$ does not converge strongly

CONTRADICTION: $(\bar{v}, \bar{u}, \bar{q})$ cannot be a point of continuity of $\text{id} : (X, d) \rightarrow (X, \| \cdot \|_{L^1})$