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Differentiability and Dimension of Some Fractal Fourier Series

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1. INTRODUCTION

Taking into account that the frequencies of Fourier expansions are integers it is not surprising that some arithmetical results play an important role in many theorems and examples in harmonic analysis. This relation is even clearer from the historical point of view and Hardy and Littlewood can be considered at the same time as founders of a substantial part of harmonic analysis and analytic number theory. For instance, in [Ha-Li] they considered

$$S_\alpha = \sum_{n=1}^{\infty} \frac{e(n^2 x)}{n^\alpha} \qquad \text{where} \quad e(t) = e^{2\pi i t}$$

and proved, using diophantine techniques, that $\operatorname{Re} S_\alpha$ and $\operatorname{Im} S_\alpha$ are not Fourier series of a L^1 function when $0 < \alpha < 1/2$ (in fact the same happens for $\alpha = 1/2$); this result was used to give an easy counterexample to some Young type inequalities. It is also known that for $1/2 < \alpha < 1$, S_α belongs to the Lebesgue space $L^p([0, 1])$ if $p < 2/(1-\alpha)$ and that S_1 is in B.M.O., the class of functions of bounded mean oscillation. Even the naive question about characterizing the values of x for which S_α converges leads to non-trivial problems in diophantine approximation and modular transformations. In [Ha-Li] it is proved that for $0 < \alpha < 1/2$

$$S_\alpha \text{ converges} \Leftrightarrow x \in \{a/q \in \mathbb{Q} : \gcd(a, q) = 1,\ 4 \mid q - 2\}.$$

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This family of series has an even longer history going back to Riemann who, according to Weierstrass [We], in 1872 thought $\operatorname{Im} S_2$ to be an example of a continuous but nowhere differentiable function. Hardy [Ha] in 1916 confirmed partially Riemann's assertion proving that $\operatorname{Im} S_2$ is not differentiable at any irrational value of x and at several kinds of rational values, but more than 50 years later J. Gerver [Ge] found infinitely many rational numbers in which the derivative exists, disproving Riemann's belief (almost a century after it was stated). As a matter of curiosity Riemann's function is differentiable exactly in the forementioned convergence set of S_α.

Riemann's function has also attracted the attention of modern authors because its relation with selfsimilarity, fractals and wavelets (see [Du] and [Ho-Tc]). The purpose of this paper is to use some tools in number theory to study several analytic properties of trigonometric series whose frequencies are squares or, more generally, kth powers. We shall be specially interested in the fractal (box-counting) dimension of their graphs and in their differentiability properties. The structure of the next sections is as follows:

In Section 2 we introduce some notation and state some known results that could be unfamiliar for non number theorists.

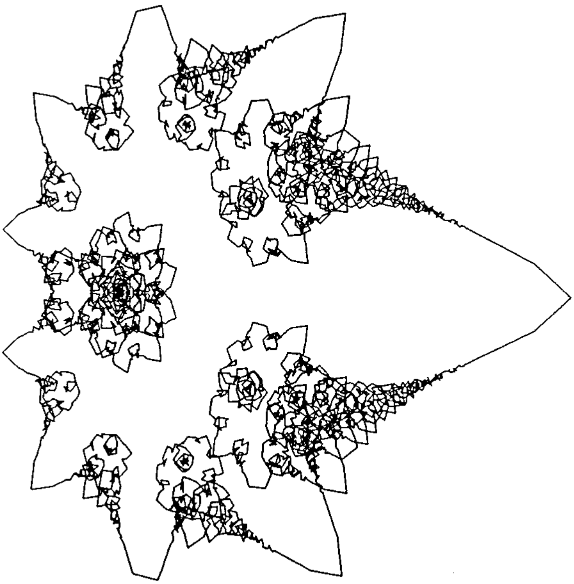
In Section 3 we estimate the fractal dimension of the graphs of trigonometric series having frequencies in the kth powers. This estimation is sharp even under mild conditions over the coefficients, extending the results of [Ch-Co]. In particular, it is obtained that the graph of these series has a fractal behavior (non-integral dimension).

In Section 4 we study the differentiability properties of

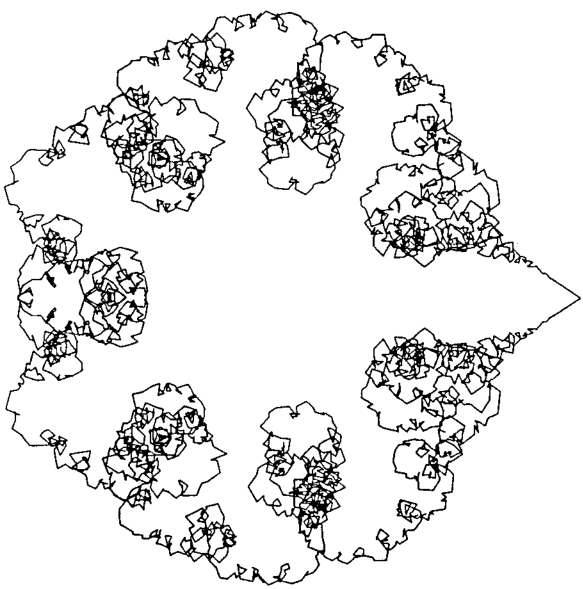
$$\sum \frac{e(n^k x)}{n^k},$$

which is an analog of Riemann's function, and the Lipschitz order of some closely related series. We get, among other things, a complete characterization of the differentiability at rational values.

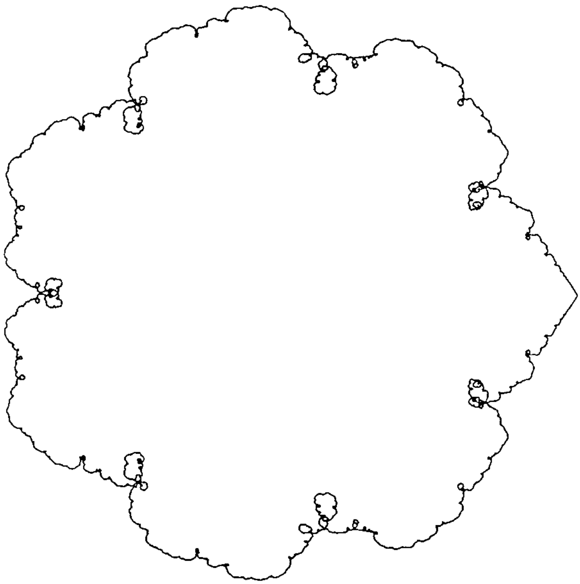
Finally in Section 5 we present a collection of computer graphics to illustrate the theoretical results obtained in previous sections. These pictures can be composed to obtain a collection of fractal-like sets in the plane. For the sake of curiosity we present here some examples which were obtained in the Seismic Park laboratory of Professor D. Córdoba.



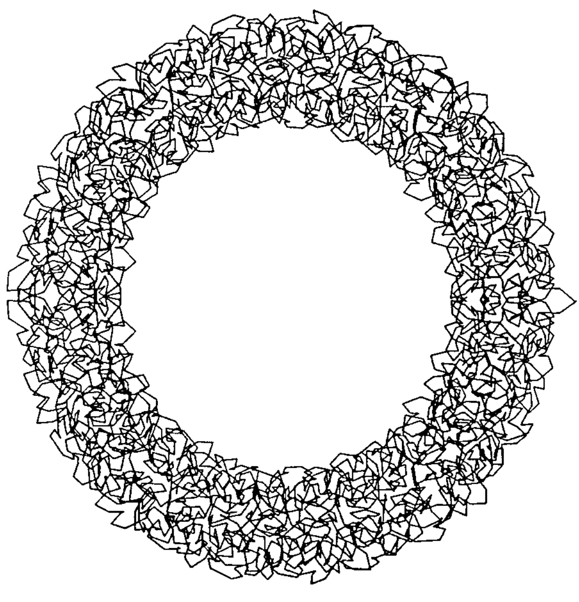
Scheme 1. $\sigma_1(t) = \left(\sum \frac{\sin(n^2 t)}{n^{3/2}}, \sum \frac{\cos(n^2 t)}{n^{3/2}} \right).$



Scheme 2. $\sigma_2(t) = \left(\sum \frac{\sin(n^3 t)}{n^2}, \sum \frac{\cos(n^3 t)}{n^2} \right).$



Scheme 3. $\sigma_3(t) = \left(\sum \frac{\sin(n^3 t)}{n^3}, \sum \frac{\cos(n^3 t)}{n^3} \right).$



Scheme 4. $\sigma_4(t) = \left(\sum \frac{\sin(n^5 t)}{n^{5/2}}, \sum \frac{\cos(n^5 t)}{n^{5/2}} \right).$

2. NOTATION AND AUXILIARY RESULTS

As we mentioned before we shall write $e(t)$ instead of $e^{2\pi it}$. We shall also use Vinogradov's symbols $f \ll g$ and $f \gg g$ meaning $|f| \leqslant C\,|g|$ and $|f| \geqslant C\,|g|$ for a certain constant, C; and Landau's notation $f = O(g)$ meaning $0 \leqslant |f|/g < C$.

The interval $[0, 1)$ can be written as the disjoint union of the intervals $I_j = [(j-1)\,h, jh)$ with $j = 1, 2, ..., N$ and $h = N^{-1}$. The number of boxes of edge h needed to cover the graph of a continuous function $f\colon [0, 1) \to \mathbb{C}$ is

$$A_h = \sum_j h^{-1} \sup_{x,\, y \in I_j} |F(x) - F(y)| + O(h^{-1}).$$

The fractal dimension (also named box-counting dimension or Minkowski dimension) of the graph of f is defined as the following limit

$$\dim(f) = -\lim_{h \to 0^+} \frac{\log A_h}{\log h}.$$

This definition only makes sense when the limit exists, by this reason it is convenient to define the lower and upper fractal dimension, $\underline{\dim}(f)$ and $\overline{\dim}(f)$, in which lim is replaced by $\underline{\lim}$ and $\overline{\lim}$, respectively.

In Section 4 we shall use $\Lambda_\beta(x)$ to denote the set of functions satisfying a Lipschitz condition of order β at x, i.e. the functions, f, such that

$$|f(x+h) - f(x)| < C\,|h|^\beta$$

holds for a certain constant, C, and small enough values of h.

As usually, we shall denote by ε an arbitrary small positive constant, not necessarily always the same. On the other hand, k will represent an integer $k > 1$.

Now we recall some results from number theory.

The proof of many interesting propositions in analytic number theory can be sometimes reduced to get a non-trivial upper bound for an exponential sum. The first method to estimate these kind of sums was created in 1916 by Weyl and it is completely based on the following lemma (see [Va]).

LEMMA (Weyl's inequality). *If P is a polynomial of degree k and a/q is an irreducible fraction such that the leading coefficient of P, say A, satisfies $|A - a/q| \leqslant q^{-2}$, then*

$$\sum_{n \leqslant N} e(P(n)) \ll_\varepsilon (Nq^{-1/K} + N^{1-1/K} + N^{1-k/K} q^{1/K})\, N^\varepsilon$$

for $K = 2^{k-1}$ and every $\varepsilon > 0$.

Using a method due to I. M. Vinogradov it is possible to improve Weyl's inequality for large values of k. There are several kinds of results obtainable by Vinogradov's method, the following one is taken from [El-Me] (see Th. 10.1) after some simplifications in the statement.

LEMMA. *Let P, k, A and q as in Weyl's inequality, then for $k \geqslant 9$ and $N \leqslant q \leqslant N^{k-2}$ we have the estimate*

$$\sum_{n \leqslant N} e(P(n)) \ll_k N^{1-\delta}$$

where $\delta = (9k^2 \log k)^{-1}$.

In some contexts (e.g. studying sums of primes) exponential sums appear with coefficients that are only controlled in average and so partial summation is not applicable. Large sieve inequality allows to obtain cancellation in some cases if we have a large enough number of these exponential sums. There are several formulations of the large sieve inequality depending on the context, perhaps the simplest is (see [Bo]).

LEMMA (Large sieve inequality). *Let $\delta > 0$ and $\{t_\nu\}_{\nu=1}^{R}$ be such that $0 \leqslant t_1 < t_2 < \cdots < t_R < 1 - \delta$ and $t_{\nu+1} - t_\nu > \delta$, then*

$$\sum_{\nu=1}^{R} \left| \sum_{n \leqslant N} a_n e(n t_\nu) \right|^2 \leqslant (N + \delta^{-1}) \sum_{n \leqslant N} |a_n|^2$$

for arbitrary complex numbers a_n.

Finally, we shall denote $(\mathbb{Z}/p^\gamma\mathbb{Z})^*$ the group of units of the ring $\mathbb{Z}/p^\gamma\mathbb{Z}$. Its order is given by the Euler totient function $\phi(p^\gamma) = p^{\gamma-1}(p-1)$. It can be proved that $(\mathbb{Z}/p^\gamma\mathbb{Z})^*$ is isomorphic to the Galois group $\mathscr{G}(\mathbb{Q}(\zeta)/\mathbb{Q})$ with $\zeta = e(1/p^\gamma)$. In particular, $\mathscr{G}(\mathbb{Q}(\zeta)/\mathbb{Q}(\zeta^p))$ is cyclic of order p.

3. FRACTAL DIMENSION

In this section we consider the family of functions

$$F_{\alpha,k}(x) = \sum_{n=1}^{\infty} \frac{c_n e(n^k x)}{n^\alpha} \qquad \text{with} \quad 0 < \underline{\lim}\, c_n \leqslant \overline{\lim}\, c_n < \infty,$$

and we compute the fractal dimension of their graphs.

THEOREM 3.1. *If* $(k+1)/2 \leqslant \alpha \leqslant k+1/2$, *then we have*

$$\dim(F_{\alpha,k}) = 2 + \frac{1}{2k} - \frac{\alpha}{k}.$$

Remark. Note that this result implies that the fractal dimension of $F_{\alpha,k}$ is non-integral for $(k+1)/2 \leqslant \alpha < k+1/2$. On the other hand, if $\alpha > k+1/2$, $F_{\alpha,k}$ is absolutely continuous (note that $F'_{\alpha,k} \in L^2$) and the fractal dimension is trivially one. The range can be completed to $1 < \alpha \leqslant k+1/2$ under the conjectural bound $\|\sum_{n \leqslant N} c_n e(n^k x)\|_{2k} = O(N^{1/2+\varepsilon})$ which is true for $k=2$.

Proof. Firstly we shall prove the upper bound

$$\overline{\dim}(F_{\alpha,k}) \leqslant 2 + \frac{1}{2k} - \frac{\alpha}{k}. \tag{3.1}$$

Given $x, y \in I$ where I is a closed interval of length h, we separate the terms of $F_{\alpha,k}$ with $n > h^{-1/(k-1)}$ and we apply then mean value theorem to those with $n \leqslant h^{-1/k}$. Then we obtain

$$|F_{\alpha,k}(x) - F_{\alpha,k}(y)| \ll h\,|S_1(\xi)| + |S_2(x)| + |S_2(y)| + h^{(\alpha-1)/(k-1)}$$

where

$$S_1(\xi) = \sum_{n \leqslant h^{-1/k}} c_n n^{k-\alpha} e(n^k \xi), \qquad S_2(t) = \sum_{h^{-1/k} < n \leqslant h^{-1/(k-1)}} \frac{c_n e(n^k t)}{n^\alpha}$$

and $\xi \in I$. Taking $I = I_j$ (with I_j as in the definition of fractal dimension) and separating the contribution of odd and even j's to A_h, we have

$$A_h \ll \sum_{\nu} |S_1(t_\nu)| + h^{-1} \sum_{\nu} |S_2(t'_\nu)| + h^{(\alpha-2k+1)/(k-1)} \tag{3.2}$$

where $h < t_{\nu+1} - t_\nu,\ t'_{\nu+1} - t'_\nu < 3h$.

Subdividing S_2 into dyadic intervals we get

$$S_2(t'_\nu) \ll \sum_{M} \left| \sum_{M < n \leqslant 2M} \frac{c_n e(n^k t'_\nu)}{n^\alpha} \right|$$

where $M = 2^m h^{-1/k}$ with $h^{-1/k} \leqslant M < h^{-1/(k-1)}$. A further subdivision of each dyadic interval $(M, 2M]$ into intervals of length $h^{-1} M^{1-k}$, we find M_1 and M_2, for each M, with $M < M_1 < M_2 \leqslant 2M$ and $M_2 < M_1 + h^{-1} M^{1-k}$ such that

$$\sum_{\nu} |S_2(t'_\nu)| \ll \sum_{M} \frac{M}{h^{-1} M^{1-k}} \sum_{\nu} \left| \sum_{M_1 < n \leqslant M_2} \frac{c_n e(n^k t'_\nu)}{n^\alpha} \right|.$$

Let us denote $S_{2M}(t'_\nu)$ the innermost sum, then by (3.2) and Cauchy's inequality we obtain

$$A_h^2 \ll h^{-1} \sum_\nu |S_1(t_\nu)|^2 + h^{-1} \sum_M M^{2k+\varepsilon} \sum_\nu |S_{2M}(t'_\nu)|^2 + h^{(2\alpha-4k+2)/(k-1)}. \tag{3.3}$$

Large sieve inequality implies

$$\sum_\nu |S_1(t_\nu)|^2 \ll (h^{-1}+h^{-1}) \sum_{n \leqslant h^{-1/k}} |c_n|^2 \, n^{2k-2\alpha} \ll h^{-3+(2\alpha-1)/k-\varepsilon}$$

and

$$\sum_\nu |S_{2M}(t'_\nu)|^2 \ll (h^{-1}+h^{-1}) \sum_{M_1 < n < M_2} |c_n|^2 \, M^{-2\alpha} \ll h^{-2} M^{1-k-2\alpha}.$$

Therefore, substituting in (3.3)

$$A_h^2 \ll h^{-4+(2\alpha-1)/k-\varepsilon}$$

and (3.1) follows.

Now we shall prove the lower bound

$$\underline{\dim}(F_{\alpha,k}) \geqslant 2 + \frac{1}{2k} - \frac{\alpha}{k}. \tag{3.4}$$

Let $f(x) = F_{\alpha,k}(x+h) - F_{\alpha,k}(x)$, by the definition of A_h and Hölder inequality

$$A_h \geqslant h^{-2} \, \|f\|_1 \geqslant h^{-2} \, \|f\|_2^3 / \|f\|_4^2. \tag{3.5}$$

Parseval's identity proves

$$\|f\|_2^2 = \sum_n |c_n|^2 \, \frac{|1-e(n^k h)|^2}{n^{2\alpha}}.$$

Considering only the contribution of $1/4h < n^k < 3/4h$, we obtain

$$\|f\|_2^2 \gg h^{(2\alpha-1)/k}.$$

On the other hand, subdividing into dyadic intervals we have that for every $\varepsilon > 0$

$$\|f\|_4^4 \ll \sup_N N^\varepsilon \left\| \sum_{N \leqslant n < 2N} c_n \frac{1-e(n^k h)}{n^\alpha} e(n^k x) \right\|_4^4;$$

hence, using again Parseval's identity

$$\|f\|_4^4 \ll \sup_N N^{\varepsilon-4\alpha} \sum_{N\leqslant n<2N} \sum_{N\leqslant m<2N} |r_k(n^k+m^k)|^2 \, |c_n|^2 \times |c_m|^2 \, |1-e(n^k h)|^2 \, |1-e(m^k h)|^2$$

where r_k denotes the number of representations as a sum of two kth-powers. It holds that $r_k(l) \ll l^\varepsilon$ (because $l=m^k+n^k$ implies that $m+n$ divides l if k is odd or $m^{k/2}+in^{k/2}$ divides l in $\mathbb{Z}[i]$ if k is even, and the number of divisors grows less than any positive power), then

$$\|f\|_4^4 \ll \sup_N N^{\varepsilon-4\alpha} \Big(\sum_{N\leqslant n<2N} |c_n|^2 \, |1-e(n^k h)|^2 \Big)^2 \ll \sup_N N^{\varepsilon-4\alpha} \min(N^2, N^{2+2k} h^{-2}) \ll h^{(4\alpha-2)/k-\varepsilon}.$$

Substituting the bounds for $\|f\|_2$ and $\|f\|_4$ in (3.5) we conclude

$$A_h \gg h^{-2-1/2k+\alpha/k+\varepsilon}$$

for every $\varepsilon>0$, which implies (3.4). ∎

4. LIPSCHITZ ORDER AND DIFFERENTIABILITY

It is known (see [Ha] and [Du]) that the Lipschitz order of $F_{2,2}$ with $c_n=1$ at x depends on diophantine properties of x. Analogous results are more difficult to obtain for $F_{\alpha,k}$ with $k>2$ among other things, because usually $|x-a/q|<q^{-k}$ does not hold for infinitely many a/q; in the terminology of the circle method (see [Va]) "major arcs" are very short. We shall firstly use Weyl's inequality to prove.

THEOREM 4.1. *If $\{c_n\}_{n=1}^\infty$ is a bounded monotonic sequence and x is irrational whose continuous fraction has bounded partial quotients, then for $1<\alpha\leqslant k+1/2$ we have*

$$F_{\alpha,k} \in \Lambda_{\beta'}(x) \qquad \text{for every} \quad \beta'<\beta=\frac{\alpha-1+2^{1-k}}{k}.$$

Remark. Note that the condition of the theorem is fulfilled by all quadratic irrationals. Several other results can be proved, using the same arguments, with different information about the behavior of the coefficients of the continuous fraction.

Furthermore, one can check that the previous result is sharp for $k=2$ (compare with [Du]), i.e. that $F_{\alpha,2} \notin \Lambda_{\beta'}(x)$ for any $\beta' > \beta$. Nevertheless, for large values of k we obtain the following improvement

THEOREM 4.2. *Let $\{c_n\}$ and x be as before, then there exists an absolute constant, $3 \leqslant C \leqslant 9$, such that for $1+(k^3 \log k)^{-1} \leqslant \alpha \leqslant k+1/2$*

$$F_{\alpha,k} \in \Lambda_\beta(x) \qquad \textit{with} \quad \beta = \frac{\alpha-1}{k} + \frac{1}{Ck^3 \log k}.$$

We specialize now the family of functions of the previous section focusing in

$$f_k(x) = \sum_n \frac{e(n^k x)}{n^k},$$

which constitute an analog of Riemann's function. By the reasons mentioned before, when $k>2$ it seems very difficult to perform a sharp analysis of the differentiability at irrational points. In our next result we give a complete characterization of the differentiability at rational points. The proof will follow the lines of [Ge] when $k=2$ but substantially simplified and generalized.

THEOREM 4.3. *Let $a/q \in \mathbb{Q}$ be an irreducible fraction and*

$$S(a/q) = \sum_{n=1}^{q} e(an^k/q).$$

Then f_k is differentiable at a/q if and only if $S(a/q)=0$. Moreover, in this case,

$$f'_k(a/q) = -\frac{2\pi i}{q} \sum_{n=1}^{q} ne(an^k/q).$$

From the well known evaluation of Gauss'sums (see for instance [Da]) it follows at once a result proved in [Du].

COROLLARY 4.4. *If a/q is an irreducible fraction, f_2 is differentiable at a/q if and only if $q \equiv 2(4)$.*

Remark. In fact it is possible to deduce from the theorem that $f_2'(a/q) = -\pi i$ when it exists, because

$$f_2'(a/q) = -\frac{2\pi i}{q}\left(q/2 + \sum_{n \neq q,\, q/2} ne(an^2/q)\right)$$

and the summation vanishes for $q = 4m+2$ and a odd due to the fact that the terms can be arranged four by four in the following manner

$$ne(an^2/q) + (q/2 - n)\, e(a(q/2-n)^2/q) + (q/2+n)\, e(a(q/2+n)^2/q) \\ + (q-n)\, e(a(q-n)^2/q) = 0.$$

In general, to decide the vanishing of $S(a/q)$ it is not needed to carry out the calculation thanks to the following lemmas.

LEMMA 4.5. *If a/q is an irreducible fraction and q factorizes as $q = p_1^{\gamma_1} p_2^{\gamma_2} \cdots p_r^{\gamma_r}$ where p_j are different prime numbers and $\gamma_j > 0$, then $S(a/q) = 0$ if and only if $S(1/p_j^{\gamma_j}) = 0$ for some $1 \leqslant j \leqslant r$.*

LEMMA 4.6. *If p is prime then $S(1/p) = 0$ if and only if* $\gcd(k, p-1) = 1$.

LEMMA 4.7. *If p is prime and $p \nmid k$ then $S(1/p^\gamma) = 0$ if and only if* $\gcd(k, p-1) = 1$ *and $k \mid \gamma - 1$.*

From these lemmas we deduce at once the following.

COROLLARY 4.8. *If a/q is an irreducible fraction and q is squarefree, f_k is differentiable at a/q if and only if there is a prime factor of q, say p, such that* $\gcd(k, p-1) = 1$.

COROLLARY 4.9. *If* $\gcd(k, q) = 1$ *and a/q is an irreducible fraction then f_k is differentiable at a/q if and only if there exists $p^\gamma \mid q$, $p^{\gamma+1} \nmid q$ with p prime such that* $\gcd(k, p-1) = 1$ *and $k \mid \gamma - 1$.*

Proof of Theorem 4.1. By the mean value theorem

$$F_{\alpha,k}(x+h) - F_{\alpha,k}(x) = 2\pi i h \sum_{n \leqslant h^{-1/k}} c_n n^{k-\alpha} e(n^k \xi) \\ + \sum_{n > h^{-1/k}} c_n \frac{e(n^k(x+h))}{n^\alpha} - \sum_{n > h^{-1/k}} c_n \frac{e(n^k x)}{n^\alpha} \qquad (4.1)$$

where $\xi \in [x, x+h]$ (we assume $h > 0$, the case $h < 0$ is similar).

By a well known theorem due to Hurwitz (see Th. 195 of [Ha-Wr]), if p_n/q_n is the nth convergent of the continuous fraction of x, $|x - p_{n+j}/q_{n+j}| < 5^{-1/2} q_{n+j}^{-2}$ holds for $j = 0$, 1 or 2. On the other hand, q_{n+1}/q_n is bounded (because so are the coefficients of the continuous fraction), then we can choose q_n such that $q_n^{-2} \ll h < q_n^{-2}$ and by Weyl's inequality we get

$$\sum_{n \leqslant N} e(n^k t) \ll_{\varepsilon} (N h^{1/2K} + N^{1-1/K} + N^{1-k/K} h^{-1/2K}) \, N^{\varepsilon} \tag{4.2}$$

for every $\varepsilon > 0$ and $t \in [x + h, x]$.

By partial summation in (4.1), (4.2) implies

$$\begin{aligned} &|F_{\alpha,k}(x+h) - F_{\alpha,k}(x)| \\ &\qquad \ll h \cdot h^{\alpha/k - 1} \cdot (h^{-1/k + 1/2K} + h^{-1/k + 1/kK} + h^{-1/k + 1/2K}) \, h^{-\varepsilon} \end{aligned}$$

for every $\varepsilon > 0$, which proves the result. ▮

Proof of Theorem 4.2. We can assume $k \geqslant 9$ because otherwise Theorem 4.1 gives a stronger result. Proceeding as in the previous proof but separating the contribution of $n \gg h^{-1/2}$, we have that $F_{\alpha,k}(x+h) - F_{\alpha,k}(x)$ equals

$$\begin{aligned} &2\pi i h \sum_{n \leqslant h^{-1/k}} c_n n^{k-\alpha} e(n^k \xi) + \sum_{h^{-1/k} < n \ll h^{-1/2}} c_n \frac{e(n^k(x+h))}{n^{\alpha}} \\ &\qquad - \sum_{h^{-1/k} < n \ll h^{-1/2}} c_n \frac{e(n^k x)}{n^{\alpha}} + O(h^{(\alpha-1)/2}). \end{aligned}$$

Note that $(\alpha - 1)/2 > (\alpha - 1)/k + (Ck^3 \log k)^{-1}$, hence the result follows by partial summation if we prove the bound

$$\sum_{n \leqslant N} e(n^k t) \ll N^{1-\delta}$$

with $\delta = (9k^2 \log k)^{-1}$ for $1 \leqslant N \ll h^{-1/2}$ and $t \in [x, x+h]$. Proceeding as before, we can always find a convergent, p_n/q_n of the continuous fraction of x such that $q_n < h^{-1/2}$ and $N < q_n \ll N$, so we are in the hypothesis of the lemma quoted in Section 2. ▮

Proof of Theorem 4.3. $(\Rightarrow)$ If f_k is differentiable at a/q, then

$$\lim_{r\to 1^-} \frac{\partial}{\partial\alpha}\bigg|_{\alpha=a/q} \int_0^1 f(t)\, P_r(\alpha - t)\, dt = f'_k(a/q)$$

where $P_r(u)$ is the Poisson kernel in $\mathbb{R}/\mathbb{Z}$ (see III.7.6 of [Zy]). Hence

$$2\pi i \lim_{r\to 1^-} (S_1(r) + S_2(r)) = f'_k(a/q) \tag{4.3}$$

where

$$S_1(r) = \sum_{n=1}^{\infty} \left(e(an^k/q) - \frac{1}{q}\, S(a/q) \right) r^{n^k}, \qquad S_2(r) = \frac{1}{q}\, S(a/q) \sum_{n=1}^{\infty} r^{n^k}.$$

Note that since the sum of q consecutives coefficients of r^{n^k} in $S_1(r)$ vanishes, then the sequence of coefficients is summable by Cesàro means, in particular $\lim_{r\to 1^-} S_1(r) < \infty$ (see III.1.33 of [Zy]). Hence it is clear that (4.3) can only hold if $S(a/q) = 0$.

As a byproduct of the proof we get that $f'_k(a/q)$, if exists, coincides with the limit of the Cesàro sums of the sequence $a_n = 2\pi i e(an^k/q)$, i.e.

$$f'_k(a/q) = -\frac{2\pi i}{q} \sum_{n=1}^{q} n e(an^k/q).$$

$(\Leftarrow)$ By partial summation $f_k(a/q+h) - f_k(a/q)$ equals, up to a constant,

$$\sum_n \left(\frac{e(n^k h)}{n^k} - \frac{e((n+1)^k h)}{(n+1)^k} \right) a_n \qquad \text{where} \quad a_n = \sum_{m=1}^{n} e(am^k/q).$$

As $S(a/q) = 0$, $a_n = a_m$ for $n \equiv m(q)$. Hence the differentiability of f_k at a/q follows from that at $h = 0$ of the functions $g_1, g_2, g_3, ..., g_q$ defined by

$$g_j(h) = \sum_{n\equiv j(q)} \left(\frac{e(n^k h)}{n^k} - \frac{e((n+1)^k\, h)}{(n+1)^k} \right).$$

Let us write g_j as

$$g_j(h) = G_{1j}(h) + G_{2j}(h) + G_{3j}(h)$$

where

$$G_{1j}(h)=\sum_{\substack{n\equiv j(q)\\ n\leqslant N}}\left(\frac{q-1}{q}\frac{e(n^kh)}{n^k}-\frac{e((n+1)^k\,h)}{(n+1)^k}+\frac{e((n+q)^k\,h)}{q(n+q)^k}\right),$$

$$G_{2j}(h)=\sum_{\substack{n\equiv j(q)\\ n\leqslant N}}\left(\frac{e(n^kh)}{n^k}-\frac{e((n+q)^k\,h)}{(n+q)^k}\right)\qquad\text{and}$$

$$G_{3j}(h)=\sum_{\substack{n\equiv j(q)\\ n> N}}\left(\frac{e(n^kh)}{n^k}-\frac{e((n+1)^k\,h)}{(n+1)^k}\right)$$

with $N=\varepsilon h^{-1/(k-1)}$ and ε to be chosen later.

Weyl's inequality implies (note that if H is the nearest integer to h^{-1} then $|h-1/H|<1/H^2$)

$$G_{3j}(h)\ll(\varepsilon^{1-k}h^{1+1/K}+\varepsilon^{1-k-k/K}h^{1+1/K(k-1)})(\varepsilon h)^{-\eta}\tag{4.4}$$

for η arbitrarily small. On the other hand, $G_{2j}(h)$ telescopes and trivially

$$G_{2j}(h)=e(j^kh)/qj^k+O(\varepsilon^{-k}h^{1+1/(k-1)}).\tag{4.5}$$

Finally, we shall treat $G_{1j}(h)$. Taylor expansion proves that for $n\leqslant N$

$$\frac{q-1}{qn^k}-\frac{e(((n+1)^k-n^k)\,h)}{(n+1)^k}+\frac{e(((n+q)^k-n^k)\,h)}{q(n+q)^k}=C_n(h)+O(n^{k-2}h^2)$$

where $C_n=a_n+b_nh$ with $a_n\ll n^{-2-k}$, $b_n\ll n^{-2}$. Hence completing the sum of $C_n(h)$

$$\begin{aligned}G_{1j}(h)=\sum_{n\equiv J(q)}C_n(h)\,e(n^kh)&+O(\varepsilon h)+O(\varepsilon^{-1-k}h^{1+2/(k-1)})\\&+O(\varepsilon^{-1}h^{1+1/(k-1)}).\end{aligned}\tag{4.6}$$

Choosing ε to be a small enough power of h, we conclude from (4.4), (4.5) and (4.6) that $g_j(h)$ is differentiable at $h=0$. ∎

Proof of Lemma 4.5. As $\gcd(p,q/p)=1$, an application of Euclid's algorithm proves (see Lemma 2.10 of [Va])

$$S(a/q)=S(a_1/p_1^{\gamma_1})\cdot S(a_2/p_2^{\gamma_2})\cdot\;\cdots\;\cdot S(a_r/p_r^{\gamma_r})$$

for some $a_1,a_2,\ldots,a_r$ with $\gcd(a_j,p_j)=1$, $1\leqslant j\leqslant r$. Finally, applying a suitable automorphism of the Galois group of $\mathbb{Q}(e(1/p_j^{\gamma_j}))/\mathbb{Q}$ it is deduced that $S(a_j/p_r^{\gamma_j})$ vanishes if and only if $S(1/p_r^{\gamma_j})$ vanishes. ∎

Proof of Lemma 4.6. first of all, note that $S(1/p)=0$ if and only if n^k runs over all of the residue classes when $1 \leqslant n \leqslant p$, because otherwise $S(1/p)$ would be a linear combination of less than p distinct p-roots of unity, which is non zero (because the cyclotomic polynomial $t^{p-1}+t^{p-2}+\cdots+t+1$ is irreducible). Hence, defining the group homomorphism

$$\phi\colon (\mathbb{Z}/p\mathbb{Z})^* \to (\mathbb{Z}/p\mathbb{Z})^*$$

$$x \to x^k$$

it is enough to prove

$$\phi \text{ is injective} \Leftrightarrow \gcd(k, p-1)=1.$$

But note that $p-1$ is the order of $(\mathbb{Z}/p\mathbb{Z})^*$, so if $\gcd(k, p-1)=d$ all the group elements with order dividing d are in $\operatorname{Ker}\phi$ (and the converse is also true). ∎

Proof of Lemma 4.7. We can assume $\gamma>1$ because otherwise the result is covered by Lemma 4.6. By notational convenience we shall write $\zeta=e(1/p^\gamma)$ and $\zeta_j=\zeta^{p^j}$. Separating the terms of $S(1/p^\gamma)$ according the power of p dividing n, we have

$$S(1/p^\gamma)=\xi_0+\xi_1+\xi_2+\cdots+\xi_\gamma, \qquad \text{where} \quad \xi_j=\sum_{\substack{n=1\\ p\nmid n}}^{p^{\gamma-j}} \zeta_{jk}^{n^k}.$$

If J is the greatest integer less than $(\gamma-1)/k$ then $\xi_j=\phi(p^{\gamma-j})$ for $J+1<j\leqslant\gamma$ and $\xi_{J+1}=\phi(p^{\gamma-J-1})$ or $\xi_{J+1}=p^{\gamma-J-2}(S(1/p)-1)$ depending on $k\mid\gamma-1$ or $k\nmid\gamma-1$. Hence

$$S(1/p^\gamma)=\xi_0+\xi_1+\xi_2+\cdots+\xi_J+(1-\varepsilon_0)\,p^{\gamma-J-2}S(1/p)+\varepsilon_0\,p^{\gamma-J-1} \tag{4.7}$$

where $\varepsilon_0=0$ if $k\mid\gamma-1$ and $\varepsilon_0=1$ if $k\nmid\gamma-1$.

After writing $S(1/p^\gamma)$ in this manner, we shall prove separately the two implications of the lemma.

($\Rightarrow$) The Galois group $\mathcal{G}(\mathbb{Q}(\zeta)/\mathbb{Q}(\zeta^p))$ is cyclic of order p generated by $\sigma_0\colon\zeta\to\zeta_{\gamma-1}\zeta$. A plain calculation proves $T_0\xi_0=0$ where T_0 is the "trace" function $T_0=1/p(\sigma_0+\sigma_0^2+\sigma_0^3+\cdots+\sigma_0^p)$. In the same way, considering the generator of $\mathcal{G}(\mathbb{Q}(\zeta_{jk})/\mathbb{Q}(\zeta_{jk+1}))$ given by $\sigma_j\colon\zeta_{jk}\to\zeta_{\gamma-1}\zeta_{jk}$, it holds for $0\leqslant j\leqslant J$

$$T_j\xi_j=0 \qquad \text{where} \quad T_j=\frac{1}{p}(\sigma_j+\sigma_j^2+\sigma_j^3+\cdots+\sigma_j^p).$$

Hence by (4.7) if $S=0$ we have, note that T_j fixes $\mathbb{Q}(\zeta_{jk+1})$,

$$0=(T_J \circ T_{J-1} \circ \cdots \circ T_1 \circ T_0)(S(1/p^\gamma))$$
$$=(1-\varepsilon_0)\, p^{\gamma-J-2} S(1/p)+\varepsilon_0\, p^{\gamma-J-1}.$$

Obviously, this equality requires $\varepsilon_0=0$ and $S(1/p)=0$, so $k \mid \gamma-1$ and $\gcd(k, p-1)=1$ by Lemma 4.6.

($\Leftarrow$) As it was proves in Lemma 4.6, $\gcd(k, p-1)=1$ implies that ϕ: $(\mathbb{Z}/p\mathbb{Z})^* \rightarrow (\mathbb{Z}/p\mathbb{Z})^*$ given by $\phi(x)=x^k$ is an isomorphism. If $p \nmid k$ each solution of $x^k=a \neq 0$ in $\mathbb{Z}/p\mathbb{Z}$ can be lifted, by Hensel's lemma, to a solution in $\mathbb{Z}/p^l\mathbb{Z}$, $0<l \leqslant \gamma$ (in fact to a p-adic solution). Hence $\phi(x)=x^k$ is an epimorphism, indeed an isomorphism, acting on $(\mathbb{Z}/p^l\mathbb{Z})^*$. Then we can write for $0 \leqslant j \leqslant J$

$$\xi_j=\sum_{\substack{n=1 \\ p \nmid n}}^{p^{\gamma-j}} \zeta_{jk}^n=p^{j(k-1)} \sum_{\substack{n=1 \\ p \nmid n}}^{p^{\gamma-jk}} e(n/p^{\gamma-jk}).$$

The last sum is a Ramanujan sum that can be explicitly evaluated (see for instance p. 31 of [Va]) and in this case it vanishes (note that $\gamma-jk>1$ because $j \leqslant J$) then the result follows from (4.7) and Lemma 4.6. ∎

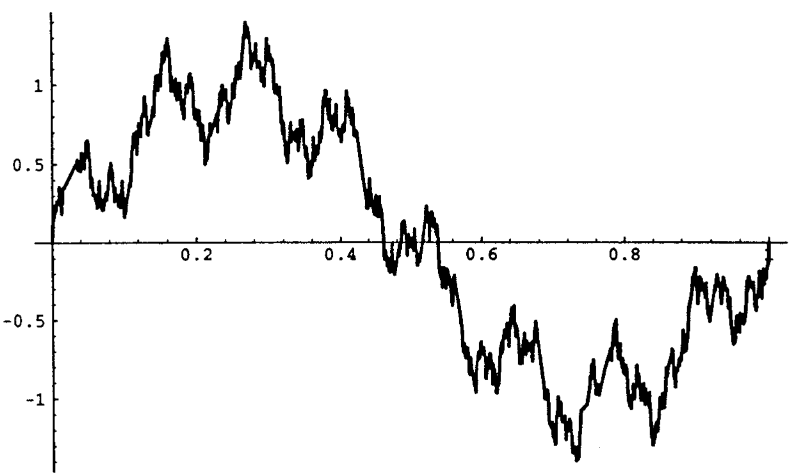


FIG. 1. Graph of Im $F_{3,2}$.

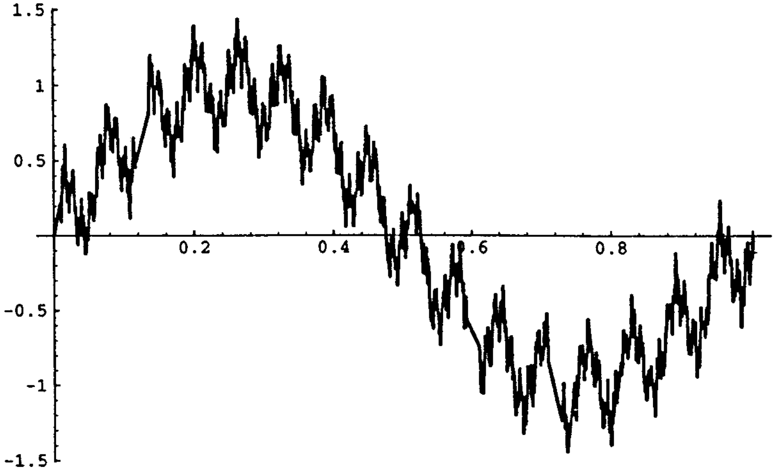


FIG. 2. Graph of $\operatorname{Im} F_{4,2}$.

5. SOME COMPUTER GRAPHICS

A global view of graphs of $\operatorname{Re} F_{\alpha,k}$ and $\operatorname{Im} F_{\alpha,k}$ reveals a chaotic behavior that increases with k when α is fixed, which agrees with Theorem 3.1. On the other hand the graphs of $\operatorname{Re} f_k$ and $\operatorname{Im} f_k$ become less chaotic as k grows due to the formula $\dim f_k = 1 + 1/2k$. This situation is reflected in Figs. 1–4 in which the sequence c_n has been chosen to be constant.

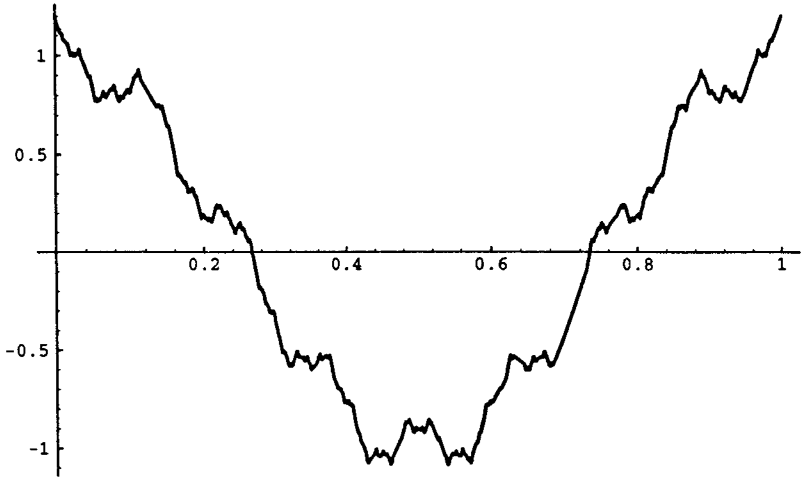


FIG. 3. Graph of $\operatorname{Re} F_{3,3}$.

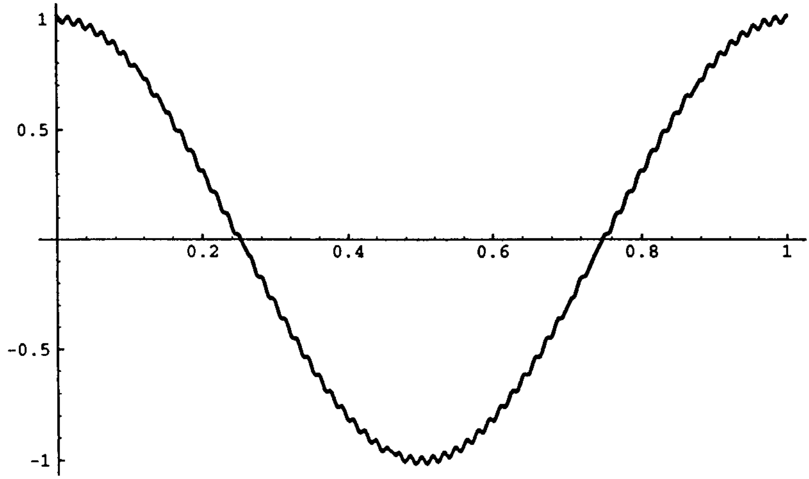


FIG. 4. Graph of Re $F_{6,6}$.

Although this global chaotic behavior, Theorem 4.3 and its corollaries imply that the functions Re f_k and Im f_k are differentiable at infinitely many rational points. If one of these functions is differentiable at $x = a/q$ then its graph can be approximated by a straight line in a small neighbourhood of x. The size of this neighbourhood decreases drastically when q or k grow. For instance, according with Corollary 4.8, Re f_3 is differentiable at $x_1 = 1/3$ and at $x_2 = 3/5$. In Figs. 4–5 it is possible to check how different are the needed ranges to visualize differentiability at x_1 and x_2 even in this

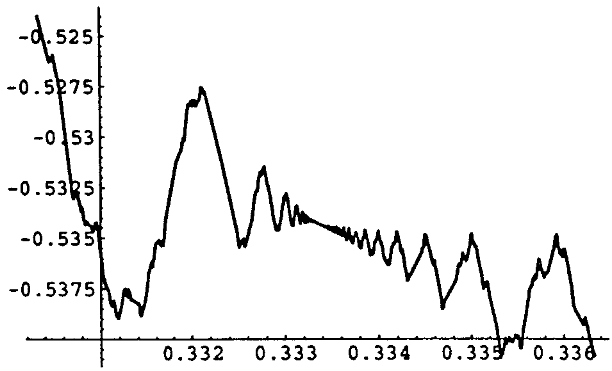


FIG. 5. Graph of Re f_3, $0.331 \leqslant x \leqslant 0.336$.

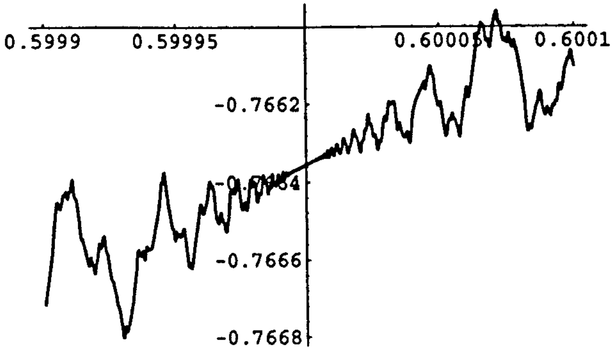


FIG. 6. Graph of $\mathrm{Re} f_3$, $0.5999 \leqslant x \leqslant 0.6001$.

case in which denominators are very close. These pictures also show the complexity of the graphs of $\mathrm{Re}\, f_k$ and $\mathrm{Im}\, f_k$, which suitably magnified reveal a quite different local-global behavior (compare Fig. 3 with 5 and 6) depending on diophantine approximations.

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